KU LEUVEN

ARENBERG DOCTORAL SCHOOL
Faculty of Science

Moment maps in multisymplectic geometry

Leyli Mammadova

Supervisor:
Prof. Dr. Marco Zambon

Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Mathematics

October 2020

Moment maps in multisymplectic geometry

Leyli MAMMADOVA

Examination committee:
Prof. Dr. Johannes Nicaise, chair
Prof. Dr. Marco Zambon, supervisor
Prof. Dr. Joeri Van der Veken
Prof. Dr. Nikolay Bobev
Prof. Dr. Simone Gutt
(Université Libre de Bruxelles)
Prof. Dr. Christian Blohmann
(Max Planck Institut für Mathematik)

Dissertation presented in partial fulfillment of the requirements for the degree of Doctor of Science (PhD): Mathematics

October 2020

Uitgegeven in eigen beheer, Leyli Mammadova, Celestijnenlaan 200B box 2400, B-3001 Leuven (Belgium)

Cover design by Ayla Valiyeva (Instagram: @koa.i.la).

To friends and kind strangers

Acknowledgments

This thesis is dedicated to all the kind people I have been fortunate to meet in my life. This is an opportunity to express my gratitude to all of them.

First and foremost, I would like to thank my advisor Marco Zambon for his mentorship, support, and patience. I thank Marco for introducing me to the world of n-plectic geometry and higher Lie theory, for teaching me how to conduct research, for posing the problems that constitute most of this thesis, and for always making time for me, no matter how busy he was. Thank you for answering all my questions, no matter how trivial and ridiculous. I could not have asked for a better advisor.

I am very grateful to the Methusalem project for giving me the opportunity to conduct research and attend conferences and workshops to broaden my scientific horizons for 4 years.

I would like to thank Leonid Ryvkin for introducing me to the problem that constituted the paper we co-authored, for sharing his mathematical insights with me, and for being my elder brother in academia.

I am grateful to the members of my examination committee, Prof. Johannes Nicaise, Prof. Joeri van der Veken, Prof. Nikolay Bobev, Prof. Simone Gutt, and Prof. Christian Blohmann, for making time to evaluate this thesis, and for their comments on the manuscript.

I would like to thank the secretariat of the Mathematics department for their help with all kinds of questions.

I thank the Geometry section at KU Leuven for the friendly environment, and for making me feel welcome from the start of my journey at KU Leuven. Anne, Charlotte, Joel, Kristof, Luc, Marilena, Reinier, Stephane, Wendy, I will miss our lunch gatherings at Alma very much.

I thank Antonio for his comments on the thesis, and for sharing his knowledge of physics with me. I thank Alfonso Garmendia for thesis-related tips and the endless supply of morale-boosting jokes and puns. I thank Ori for sharing excellent jokes that I have added to my personal joke archive. I thank Alfonso Tortorella for sharing great music and heartfelt conversations, and saving my private defense presentation. I thank Marco Usula for our conversations on the interplay between mathematics and physics, and for the pizza evenings. Karandeep, thank you for being a great office mate.

Going back to my time in Zurich, I would like to thank my friends Rena and Simon for helping me adapt to the new environment, and Margrit and Albert for welcoming me into their home. I thank Maurice Cochand for his constant support and for sharing his wisdom with me. I thank Stephan Tornier for showing incredible kindness and care during a difficult time. I would like to thank my master's thesis advisor Prof. Gian Michele Graf for introducing me to the wonderful world of theoretical physics.

I thank my friends Ayla, Rena, Parvin, Elnara, and Javid for their friendship, constant support, and valuable advice. Special thanks to Ayla for creating the beautiful cover of this book.

I thank my friend Peter Faber for believing in me, for making me believe in myself, and for being an example of integrity. I look up to you and learn from you.

I thank my family, the Mammadovs and the Böings, for their support and their faith in me. Mom and Dad, thank you for having my back and making sure I received the best education that was available. Ria and Ben, thank you for your constant encouragement and the warmth of your home.

There is a lot I owe to Michael Eichmair that words cannot express. You are an inspiration to all your students, both as a mathematician and a human being. Martine, thank you for being our friend, for your love and support.

Finally, I thank my husband Steef for a lot of things: for his emotional support, encouragement, for his help with LaTeX, for taking care of everything else just so that I could concentrate on the thesis. I am so fortunate to have you in my life.

A children's song from a Soviet cartoon that I loved as a child has the following line: "Kindness is the most important and precious, the most trusting and demanding thing in the world". I thank you all for trusting me with such a precious gift.

Abstract

This thesis focuses on various notions of moment maps in multisymplectic geometry. In particular, it compares two existing notions of Lie algebra moment maps in multisymplectic geometry, introduces and studies properties of Lie 2-algebra moment maps in multisymplectic geometry, and finally, provides examples of the above-mentioned various moment maps, including the construction of a moment map for a new class of multisymplectic manifolds, i.e., the generalization of coadjoint orbits to multisymplectic geometry.

Beknopte samenvatting

Dit proefschrift richt zich op verschillende noties van momentafbeeldingen in multisymplectische meetkunde. In het bijzonder worden twee bestaande noties van Lie-algebra-momentafbeeldingen in multisymplectische meetkunde vergeleken en de eigenschappen van Lie 2-algebra-momentafbeeldingen in multisymplectische meetkunde geïntroduceerd en bestudeerd. Ten slotte worden voorbeelden van de bovengenoemde verschillende momentafbeeldingen gegeven, inclusief de constructie van een momentafbeelding voor een nieuwe klasse van multisymplectische variëteiten, namelijk de veralgemening van coadjuncte orbieten naar multisymplectische meetkunde.

Contents

Chapter 1

Introduction

As the title suggests, this thesis is about moment maps in multisymplectic geometry. In this introduction, we will try to provide context, motivate the questions that lead to this thesis, and explain the results obtained.

Multisymplectic, or n-plectic, geometry is the generalization of symplectic geometry, the latter corresponding to $n = 1$. Symplectic geometry is the study of smooth manifolds equipped with a closed nondegenerate 2-form. It originated in the Hamiltonian formulation of classical mechanics, where the phase space of a physical system is a symplectic manifold, and the solutions of equations of motion are curves tangent to certain vector fields. The symplectic form is needed to associate such a vector field to functions on phase space (see §2.1).

Multisymplectic geometry arose from the attempts to formulate classical field theory in a similar way, first appearing in the works of W. M. Tulczyjew, J. Kijowski and W. Szczyrba in [33] and [34]. The *multiphase space* of such a theory is an n-plectic manifold, that is, a smooth manifold equipped with a closed nondegenerate $(n + 1)$-form. Just as solutions of the field equations correspond to curves in symplectic geometry, where $n = 1$, the solutions in n-plectic geometry correspond to "n-curves" ([26]), i.e., "curves parameterized by n-dimensional spacetime". The $(n + 1)$-form then associates to a function on multiphase space an n-vector field that is tangent to the n-curve that is the solution of the field equations.[1]

The modern definition of the symplectic moment map was introduced by J.

[1] Some authors choose to associate a vector field to an n-form instead, where the latter plays the form of Hamiltonian density. However, this approach seems more difficult to characterize geometrically. We thank Antonio Miti for pointing this out to us.

M. Souriau ([57]). This definition relating the Lie algebra of symmetries of a symplectic manifold to the Lie algebra of its smooth functions (called the Lie algebra of *observables*) generalized the already known examples of linear and angular momentum from classical physics (hence the name), and turned out to have many applications in both physics and mathematics (see §2.4).

The natural question of generalizing this notion to multisymplectic geometry has led to multiple definitions listed in §3.4. This thesis focuses on 2 notions of n-plectic moment maps that generalize other definitions existing in the literature.

The first one is the *homotopy moment map* introduced by M. Callies, Y. Fregier, C. L. Rogers and M. Zambon in [7]. Just as the symplectic moment map relates the Lie algebra of symmetries of a symplectic manifold to its Lie algebra of observables, the homotopy moment map relates the Lie algebra of symmetries of an n-plectic manifold to its algebra of observables. The difference is that in the case of an n-plectic manifold, its algebra of observables is not a Lie algebra, but something called an *L_∞-algebra* (see §3.3.3).

Another generalization of the symplectic moment map we consider is the *weak moment map* introduced by J. Herman in [29] and [30]. This map is obtained by ignoring the last equation in the definition of a homotopy moment map and restricting the domain to something called *the Lie kernel* (see §2.2.3). Thus, a homotopy moment map is an example of a weak moment map. However, there are situations where a weak moment map exists, but a homotopy moment map does not. We compare the two notions in §4; in particular, we investigate when the existence of a weak moment map implies the existence of a homotopy moment map.

Situations where a homotopy moment map doesn't exist lead to another phenomenon: the existence of "homotopy moment maps" for something called *central extensions*[2] of Lie algebras of symmetries (see §2.2.4, §2.4.3.3, §3.2.1.1, and §3.4.3.1). When the manifold under consideration is n-plectic, the corresponding central extension of the Lie algebra of symmetries is a *Lie n-algebra* (see §3.2.1). Thus, we naturally arrive at the notion of a "homotopy moment map" whose domain of definition is a Lie n-algebra rather than a Lie algebra. This naturally leads us to investigate more general "homotopy moment maps" from Lie n-algebras (i.e., not only the ones that are central extensions of Lie algebras) in §5.

[2]Such a situation in symplectic geometry corresponds to what is known as "classical anomaly" in mechanics.

1.0.1 Structure of the thesis

This thesis consists of an introduction and four chapters. The four chapters comprise the mathematical content of the thesis: the first two consisting mostly of background material (with the exception of the material in §3.3.1.1, §3.4.2.1), and the last two containing original results. Every chapter begins with an introduction, where a brief history of the main concepts is given.

Chapter 2 tells the story of symplectic geometry and the symplectic moment map. It provides all the necessary mathematical background: from Lie group and Lie algebra actions to definition and examples of symplectic manifolds to definition, examples, and properties of symplectic moment maps.

Chapter 3 introduces multisymplectic geometry and homotopy moment maps. The structure of Chapter 3 mirrors the structure of Chapter 2 while noting important differences between similar concepts in symplectic and multisymplectic geometry. There is an important caveat: Chapter 3 starts with an introduction to L_∞-algebras, which may give an impression that we will be looking at actions of L_∞-algebras. This is not the case: even in multisymplectic geometry, we investigate actions of ordinary Lie groups and Lie algebras. The only original material in this chapter is presented in §3.3.1.1 and §3.4.2.1.

Chapter 4 introduces weak moment maps, following [29], and compares weak moment maps and homotopy moment maps, providing illustrative examples. This chapter is based on work ([43]) done in collaboration with Leonid Ryvkin.

Chapter 5 tells the story of Lie 2-algebra moment maps. As noted before, in this setting we still consider actions of Lie groups/algebras, i.e., a Lie algebra morphism $\mathfrak{g} \to \mathfrak{X}(M)$. However, we will be interested in the lifts of this morphism to a morphism whose domain of definition is a Lie 2-algebra. This chapter is based on work ([44]) done in collaboration with my advisor Marco Zambon.

1.0.2 Results

Below we summarize the results obtained in the thesis.

Chapter 4

The contents of this chapter are based on a preprint co-authored by the author of the thesis and Leonid Ryvkin.

To introduce the main result of Chapter 4, we first give the following definitions.

Definition 1.0.1. [7, Def. 5.1] Let $\mathfrak{g} \to \mathfrak{X}(M), x \mapsto v_x$ be a Lie algebra action on an n-plectic manifold (M, ω). A *homotopy moment map* for this action is an L_∞-morphism $\{f_k\} : \mathfrak{g} \to L_\infty(M, \omega)$, such that $-\iota_{v_x}\omega = d(f_1(x))$ for all $x \in \mathfrak{g}$.

In other words, it is a collection of maps $\{f_k\}$, $1 \leq k \leq n+1$ such that for all $p \in \wedge^k \mathfrak{g}$:

$$-f_{k-1}(\delta_k(p)) = d(f_k(p)) + \zeta(k)\iota_{v_p}\omega \tag{1.1}$$

$$f_0 = f_{n+1} = 0, \tag{1.2}$$

where δ_k is the k-th Lie algebra homology differential (Definition 2.2.35), v_p is the fundamental vector field corresponding to p (Definition 2.2.20), and for $k \in \mathbb{N}$, $\zeta(k)$ is given by $\zeta(k) := -(-1)^{\frac{k(k+1)}{2}}$.

Definition 1.0.2. [30, Def. 3.11] Let $\mathfrak{g} \to \mathfrak{X}(M), x \mapsto v_x$ be a Lie algebra action on an n-plectic manifold (M, ω) by Hamiltonian vector fields. A *weak (homotopy) moment map* is a collection of linear maps $\widehat{f_k} : P_{k,\mathfrak{g}} \to \Omega^{n-k}(M)$, where $1 \leq k \leq n$, satisfying

$$d(\widehat{f_k}(p)) = -\zeta(k)\iota_{v_p}\omega$$

for $k \in 1, ..., n$ and all $p \in P_{k,\mathfrak{g}}$, where $P_{k,\mathfrak{g}}$ is the subspace of $\wedge^k \mathfrak{g}$ that consist of elements in the kernel of δ_k.

The main result of Chapter 4 is given by Theorem 4.3.3. It follows from comparing the two definitions above, that existence of a homotopy moment map implies existence of a weak moment map. Theorem 4.3.3 answers the question of whether the reverse implication holds. It turns out that it does, given the following map vanishes identically:

$$\phi : P_{n+1,\mathfrak{g}} \to C^\infty(M), \;\; p \mapsto \iota_{v_p}\omega. \tag{1.3}$$

Theorem 1.0.3. *(Thm. 4.3.3) Let (M, ω) be an n-plectic manifold, and let $\mathfrak{g}$ act on (M, ω) by preserving ω. The following statements are equivalent:*

1. *The action of $\mathfrak{g}$ on (M, ω) admits a homotopy moment map*
2. *The action of $\mathfrak{g}$ on (M, ω) admits a weak moment map and $\phi \in P^*_{n+1,\mathfrak{g}} \otimes C^\infty(M)$ given by* (1.3) *vanishes identically.*

Lemma 4.3.2 connects the map ϕ to a $(n+1)$-cocycle introduced in [7].

Since a homotopy moment map gives a weak moment map by restriction, and the existence of a weak moment map together with the vanishing of ϕ defined in (1.3) imply the existence of a homotopy moment map, a natural question is whether every weak moment map is a restriction of a homotopy moment map if $\phi = 0$. Proposition 4.4.1 answers this question negatively:

Proposition 1.0.4. *(Prop. 4.4.1) Let $\widehat{f}$ be a weak moment map, and $\phi = 0$. There exists a well-defined class $[\gamma]_{\widetilde{d}_{tot}} \in H^{n+1}(\widetilde{C})$ such that the following are equivalent:*

1. *$[\gamma]_{\widetilde{d}_{tot}} = 0$ and γ admits a primitive in $\bigoplus_{k=1}^{n} d_{\mathfrak{g}}(\Lambda^k\mathfrak{g}^*) \otimes \Omega^{n-k-1}(M)$*
2. *There exists a homotopy moment map $\widetilde{f}$, such that $\widetilde{f}|_{P_{\mathfrak{g}}} = \widehat{f}$.*

Finally, following results of [18] and [53], Proposition 4.2.4 and Theorem 4.5.6 phrase existence of weak moment maps and equivariant weak moment maps in terms of the cohomology of a certain double complex (and its subcomplex, in the case of equivariant weak moment maps).

Theorem 1.0.5. *Let $\mathfrak{g}$ act on (M, ω) by preserving ω. The action admits*

1. *a weak moment map if and only if $[\widehat{\omega}] \in H^{n+1}(\widehat{C})$*
2. *a $\mathfrak{g}$-equivariant weak moment map if and only if $[\widehat{\omega}] = 0 \in H(\widehat{C}^{\mathfrak{g}})$*

Moreover, the respective moment maps are in one-to-one correspondence with potentials of the respective cohomology classes.

Chapter 5

The contents of this chapter are based on a paper co-authored by the author of the thesis and Marco Zambon.

As mentioned before, if a homotopy moment map for a given Lie algebra $\mathfrak{g}$ acting on an n-plectic manifold (M, ω) by Hamiltonian vector fields doesn't exist, it exists for a central n-extension of $\mathfrak{g}$ ([7, Prop. 9.10]). Since central n-extensions are Lie n-algebras, this motivates us to look at "homotopy moment maps" whose domain of definition is an arbitrary Lie n-algebra. Another reason to consider more general objects than a Lie algebra $\mathfrak{g}$ as the domain is that a homotopy moment map is a morphism in the category of L_∞-algebras, and there is no a priori reason to restrict ourselves to considering maps from Lie algebras rather than general L_∞-algebras.

In Chapter 5 we consider homotopy moment maps from a Lie 2-algebra $\mathfrak{h} \oplus \mathfrak{g}$ into the Lie 2-algebra $(L_\infty(M,\omega), d, [\ ,\ ]', [\ ,\ ,\ ]')$ of observables of a 2-plectic manifold. Extending the definition of a homotopy moment map ([7, Def.5.1]), we define:

Definition 1.0.6. A *homotopy moment map* for the Lie 2-algebra $\mathfrak{h} \oplus \mathfrak{g}$ (or $\mathfrak{h} \oplus \mathfrak{g}$ *moment map* for short) is an L_∞-morphism (f_1, f_2) from $(\mathfrak{h} \oplus \mathfrak{g}, \delta, [\ ,\ ], [\ ,\ ,\ ])$ to $(L_\infty(M,\omega), d, [\ ,\ ]', [\ ,\ ,\ ]')$ such that for all $x \in \mathfrak{g}$

$$-\iota_{v_x}\omega = d(f_1(x)).$$

In §5.3 we show, following [18] and [53], that homotopy moment maps for Lie 2-algebras correspond to primitives of a certain element $\widetilde{\omega}$ in a complex (C, d_{tot}) that is constructed out of the Chevalley-Eilenberg complex of the Lie 2-algebra (see Example 3.2.17) and the de Rham complex of the manifold.

Proposition 1.0.7. *(Proposition 5.3.2) There is a bijection*

$$\{\textit{moment maps for } \mathfrak{h} \oplus \mathfrak{g}\} \cong \{\mu \in C^2 : d_{tot}\mu = \widetilde{\omega}\}.$$

Theorem 5.4.9 explicitly constructs a moment map for a given Lie 2-algebra $\mathfrak{h} \oplus \mathfrak{g}$, using primitives of a certain element ω_{3p} in the Chevalley-Eilenberg complex of $\mathfrak{h} \oplus \mathfrak{g}$ and the map γ introduced in [7].

Theorem 1.0.8. *(Thm. 5.4.9) Assume $H^1(M) = 0$. Let $\eta \in CE(L)^2$ satisfy $d_{CE(L)}\eta = \omega_{3p}$. Then*

$$\phi^\eta := \gamma \circ f$$

is a moment map for $\mathfrak{h} \oplus \mathfrak{g}$, where f is constructed out of η as in Lemma 5.4.8, and γ is given just below Proposition 5.4.3.

Moreover, Proposition 5.4.14 then shows that any other moment map for $\mathfrak{h} \oplus \mathfrak{g}$ is equivalent (in the sense made precise in Definition 5.4.13) to the one constructed using the theorem above.

Since the Chevalley-Eilenberg complex of a Lie 2-algebra $\mathfrak{h} \oplus \mathfrak{g}$ is a relatively unfamiliar and more complicated than that of a Lie algebra $\mathfrak{g}$, section §5.5.1 gives existence results in terms of Lie algebra cohomology, namely in terms of a certain 3-cocycle c_{red} in the Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in a trivial representation induced by the ternary bracket of the Lie 2-algebra:

Proposition 1.0.9. *(Prop. 5.5.5, Prop. 5.5.7) Assume $[\omega_{3p}]_{\mathfrak{g}} \neq 0$.*

i) If $[c_{red}]_{\mathfrak{g}} = 0$, then there exists no $\mathfrak{h} \oplus \mathfrak{g}$ moment map.

ii) Assume $H^1(M) = 0$. If $H^3(\mathfrak{g})$ is one-dimensional and $[c_{red}]_{\mathfrak{g}} \neq 0$, then there exists a $\mathfrak{h} \oplus \mathfrak{g}$ moment map.

Chapter 3

Finally, we briefly mention the preliminary results of sections §3.3.1.1 and §3.4.2.1.

Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. Then an orbit of arbitrary $\xi \in \mathfrak{g}^*$ under the coadjoint action of G has a symplectic structure that plays an important role in symplectic geometry and mathematical physics (see [35]). The results of §3.3.1.1 and §3.4.2.1 present the beginnings of ongoing work (joint with Marco Zambon), the aim of which is, in particular, to research generalizations of coadjoint orbits and their potential applications to multisymplectic geometry. In fact, the setting is more general: let $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ be a representation of a Lie algebra $\mathfrak{g}$, and let c be a 3-cocycle in the Lie algebra cohomology of $\mathfrak{g}$ with values in $\mathfrak{h}$. Consider the representation on $\mathfrak{h}$ of the connected, simply connected Lie group G integrating $\mathfrak{g}$, and the induced representation of G on $\mathfrak{h}^*$. Then, if a certain condition holds, there exists a G-invariant 2-plectic (Proposition 3.3.11, Proposition 3.3.13) form on the orbit σ_ξ of a point $\xi \in \mathfrak{h}^*$ defined by

$$\omega_\xi(v_x, v_y, v_z) := \xi(c(x, y, z)).$$

Let $c = d_{\mathfrak{g}} b$ for $b \in \wedge^2\mathfrak{g}^* \otimes \mathfrak{h}$ a G-invariant 2-cochain in the Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in $\mathfrak{h}$. Define a 2-form on σ_ξ by $\beta_\xi(v_x, v_y) := \xi(b(x, y))$. Then, assuming certain condition (3.20) holds, the action of $\mathfrak{g}$ on the orbit σ_ξ of $\xi \in \mathfrak{h}^*$ admits a homotopy moment map:

Proposition 1.0.10. *(Prop. 3.4.10) The G-action on (σ_ξ, ω) admits an equivariant homotopy moment map given by*

$$f_1 : \mathfrak{g} \to \Omega^1_{Ham}(\sigma_\xi), x \mapsto \iota_{v_x}\beta$$

$$f_2 : \wedge^2\mathfrak{g} \to C^\infty(\sigma_\xi), x \wedge y \mapsto -\iota_{v_y}\iota_{v_x}\beta$$

The case where $\mathfrak{h} = \wedge^2\mathfrak{g}$, and $\beta = Id|_{\wedge^2\mathfrak{g}}$ corresponds to 2-plectic coadjoint orbits.

The next step in this project would be to relax the requirements on the cocycle c and see when the action admits a moment map.

Chapter 2

Symplectic geometry

This chapter will focus on symplectic geometry. The results of this chapter are not original and are based on a number of well-known sources, including [8], [23], [22], [62], [65], and [66].

§2.2 introduces the necessary background on Lie groups and Lie algebras: in particular, Lie group and Lie algebra actions.

§2.3 gives an introduction to symplectic geometry.

§2.4 introduces symplectic moment maps, provides examples and investigates the questions of existence and uniqueness.

2.1 Introduction

Symplectic geometry is the study of manifolds equipped with a closed non-degenerate 2-form. According to [65], the first symplectic manifold was introduced by Lagrange in 1808, in his study of the motion of planets. However, the importance of a symplectic structure clearly emerged from Hamilton's work on classical mechanics and optics ([23]). Thus, symplectic geometry is the mathematical framework of Hamiltonian mechanics.

In Hamiltonian mechanics, the phase space, i.e., the space of states, of a physical system is a symplectic manifold, and time evolution of a dynamical system is a one-parameter family of symplectomorphisms. Hamilton's equations assign a dynamical system to a function on the phase space that is called a Hamiltonian function and represents the energy of the system. The symplectic form assigns an evolution vector field to the Hamiltonian.

We will illustrate this using the example of a particle of mass m moving in $\mathbb{R}^n$. Such a physical system is described by a set of position and momentum coordinates $(q_1(t), ..., q_n(t), p_1(t), ..., p_n(t))$, where the momenta are given by $p_i(t) = m\dot{q}_i(t)$. The set of all possible positions and momenta of the particle constitute the *phase space* of the particle and is given by $\mathbb{R}^{2n}$.

Let the force acting on the particle be given by[1] $F = -\nabla U(q)$. The following function on the phase space is called *the Hamiltonian* of the physical system under consideration:

$$H(p,q) := \frac{|p|^2}{2m} + U(q).$$

The *Hamilton's equations* are then given by

$$\dot{q}_i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t))$$

$$\dot{p}_i(t) = -\frac{\partial H}{\partial q_i}(q(t), p(t)).$$

If we consider the vector field $v_H := (\frac{\partial H}{\partial p_1}, ..., \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q_1}, ..., -\frac{\partial H}{\partial q_n})$, the Hamilton's equations can be rewritten as

$$(\dot{q}(t), \dot{p}(t)) = v_H(q(t), p(t)).$$

i.e., the system evolves along the vector field v_H.

The symplectic structure ω is precisely what allows to associate such an evolution vector field v_f to an arbitrary function f on the phase space via

$$-\iota_{v_f}\omega = df,$$

where ι_v denotes contraction with vector field v.

E.g., the vector field $v_H := (\frac{\partial H}{\partial p_1}, ..., \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q_1}, ..., -\frac{\partial H}{\partial q_n})$ is the evolution vector field associated to the Hamiltonian $H(p,q) := \frac{|p|^2}{2m} + U(q)$ using the canonical symplectic structure of $\mathbb{R}^{2n} : \omega = \sum_{i=1}^{n} dq_i \wedge dp_i$.

Non-degeneracy of the symplectic forms assures that this vector field is unique.

[1] Such forces are called *conservative*, and $U(q)$ is called the *potential*.

The other 2 properties of the symplectic form, i.e., skew-symmetry and closedness, have the following physical interpretations ([28]):

- Skew-symmetry: The Hamiltonian of a physical system is interpreted as its total energy. Total energy of the system is conserved during the physical evolution, i.e.,

$$\pounds_{v_H} H = dH(v_H) = -\omega(v_H, v_H) = 0.$$

- Closedness: $d\omega = 0$ forces ω to be preserved under the flow of v_H, i.e., under the evolution of the system. Indeed,

$$\begin{aligned}\pounds_{v_H}\omega &= d\iota_{v_H}\omega + \iota_{v_H} d\omega \\ &= -ddH + \iota_{v_H} d\omega \\ &= \iota_{v_H} d\omega \\ &= 0.\end{aligned}$$

Despite originating in Hamiltonian mechanics, symplectic geometry evolved into a powerful technique in both physics and mathematics. In physics many complex systems are studied using Hamiltonian techniques. Generalizing symplectic geometry to infinite-dimensional manifolds allows to study classical field theory [2] ([22]).

In mathematics and mathematical physics symplectic geometry has been instrumental in developments in algebraic geometry, partial differential equations, representation theory ([65]) and geometric quantization ([66]).

2.2 Lie group and Lie algebra actions

Lie group and Lie algebra actions are central to the material of this thesis. This section gives a brief introduction to Lie group and Lie algebra actions: §2.2.1 discusses Lie group actions and provides important examples of such; §2.2.2 does the same for Lie algebra actions. §2.2.3 introduces Lie algebra homology and cohomology. Finally, §2.2.4 discusses Lie algebra extensions, which will be important in the discussion of moment maps.

The material of this section is mostly based on the lecture notes "Introduction to Lie groups" by Zuoqin Wang ([62]) and on "Introduction to smooth manifolds"

[2] We will say more about this in Section 3.

by John M. Lee [40]. We will only consider finite-dimensional Lie groups and Lie algebras.

2.2.1 Lie group actions

We first recall definitions of a Lie group and a Lie algebra:

Definition 2.2.1. ([62, Def. 1.1, Lec. 5]) A *Lie group* G is a smooth manifold equipped with a group structure, such that the multiplication map

$$m : G \times G \to G$$

$$(g_1, g_2) \mapsto g_1 \cdot g_2$$

is smooth.

Let G be a Lie group, and let L_g denote left multiplication by $g \in G$. Let T_eG be the tangent space to G at identity $e \in G$, and let $x_e \in T_eG$. Define a vector field x_g by

$$x_g := (dL_g)(x_e),$$

where dL_g is the derivative of L_g.

Definition 2.2.2. ([62, Def. 2.1, Lec. 5]) A vector field $x \in \mathfrak{X}(G)$ on a Lie group G is called *left-invariant* if

$$(dL_g)(x_h) = x_{gh}.$$

Note that this is equivalent to $L_{g*}x = x$, where the star denotes the pushforward.

It then follows from the chain rule that the vector field $x_g := (dL_g)(x_e)$ defined above is left-invariant, i.e., any vector at T_eG determines a left-invariant vector field on G. Conversely, any left-invariant vector field is determined by its value at the identity $e \in G$, i.e., the space of left-invariant vector fields on G can be identified with T_eG. In what follows, we will often identify a left-invariant vector field x with its value at the identity.

We can now define the Lie algebra of a Lie group:

Definition 2.2.3. Let G be a Lie group. The Lie algebra $\mathfrak{g}$ of G is the set of left-invariant vector fields on G equipped with the Lie bracket of vector fields[3].

[3] Note that it follows from the naturality of the Lie bracket of vector fields, that left-invariant vector fields are closed under the Lie bracket, i.e., $L_{g*}[x, y] = [L_{g*}x, L_{g*}y] = [x, y]$.

We also give the following general definition of a Lie algebra.

Definition 2.2.4. A *Lie algebra* $\mathfrak{g}$ is a vector space together with a bilinear skew-symmetric map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following *Jacobi identity* for all $x, y, x \in \mathfrak{g}$:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Finally, we recall the following result and refer the reader to [40] for the proof.

Proposition 2.2.5. *[40, Thm. 8.44, Thm. 20.19] Let G and H be Lie groups, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their respective Lie algebras. Then, if $\phi : G \to H$ is a Lie group homomorphism, then $d_e\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

Conversely, suppose G and H are Lie groups, where G is simply connected, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their corresponding Lie algebras. Let $\psi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then there exists a unique Lie group homomorphism $\Psi : G \to H$ such that $d_e\Psi = \psi$.

We can now define Lie group actions on manifolds:

Definition 2.2.6. Let G be a Lie group, and M a manifold. A homomorphism of groups

$$\psi : G \to \mathrm{Diff}(M)$$

$$g \mapsto \psi_g$$

is called an *action* of G on M.

The action is *smooth* if the following map, called the *evaluation map*, is smooth:

$$ev_\psi : G \times M \to M$$

$$(g, m) \mapsto \psi_g(m).$$

We will often denote $\psi_g(m)$ by $g \cdot m$ or gm for the sake of convenience.

When G acts linearly on a vector space, we have the following:

Definition 2.2.7. Let G be a Lie group, and V a (finite-dimensional) vector space. A homomorphism of Lie groups

$$\psi : G \to GL(V)$$

is called a *representation* of G on V.

We will also call the vector space V a representation of G.

Example 2.2.8. Any Lie group G acts on itself by left multiplication: $\psi_g(h) := g * h$, where $g, h \in G$ and $*$ denotes the group operation.

Example 2.2.9. Any Lie group G acts on itself by conjugation: $\psi_g(h) := g * h * g^{-1}$, where $g, h \in G$ and $*$ denotes the group operation.

Example 2.2.10. Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. Denote by c_g the conjugation by element $g \in G$, i.e., $c_g(h) := ghg^{-1}$ for any $h \in G$. For each $g \in G$, denote by Ad_g the derivative of c_g at the identity element: $Ad_g = d_e c_g : T_eG \to T_eG$. Since each c_g is a Lie group homomorphism, each Ad_g is a Lie algebra homomorphism. Furthermore, since the map $g \mapsto c_g$ is a homomorphism, we get

$$Ad_{gh} = d_e c_{gh} = d_e(c_g \circ c_h) = d_{c_h(e)} c_g \circ d_e c_h = d_e c_g \circ d_e c_h = Ad_g \circ Ad_h,$$

i.e., the map

$$Ad : G \to Aut(\mathfrak{g}) = GL(\mathfrak{g})$$

$$g \mapsto Ad_g$$

gives an action of G on $\mathfrak{g}$ called the *adjoint action* or the *adjoint representation* of G.

Example 2.2.11. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. The *coadjoint action* Ad^* of G on $\mathfrak{g}^*$ is given by

$$\langle Ad_g^* \xi, x \rangle = \langle \xi, Ad_{g^{-1}}(x) \rangle,$$

for $\xi \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$. We have

$$\begin{aligned} \langle Ad_{gh}^* \xi, x \rangle &= \langle \xi, Ad_{(gh)^{-1}} x \rangle \\ &= \langle \xi, Ad_{h^{-1}g^{-1}} x \rangle \\ &= \langle \xi, Ad_{h^{-1}} Ad_{g^{-1}} x \rangle \\ &= \langle Ad_g^* Ad_h^* \xi, x \rangle, \end{aligned}$$

i.e., Ad^* indeed defines an action.

Before proceeding further, we recall the following facts from the theory of Lie groups:

Proposition 2.2.12. ***(Derivative of Ad)*** *Let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra. Let $Ad : G \to GL(\mathfrak{g})$ be the adjoint representation of G. Then $d_e Ad = ad$ is a Lie algebra representation*

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

$$x \mapsto [x,\ ]$$

Proof. See the proof of [40, Theorem 20.27, §20]. □

Definition 2.2.13. [62, Def. 2.1, Lec. 6] Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. The *exponential map* of G is defined as

$$exp : \mathfrak{g} \to G,\ exp(x) = \gamma(1),$$

for any $x \in \mathfrak{g}$, where γ is the integral curve of the left-invariant vector field x starting at identity $e \in G$.

Proposition 2.2.14. ***(Naturality of the exponential map)*** *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\phi : G \to H$ be a Lie group homomorphism. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d_e\phi} & \mathfrak{h} \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} \\ G & \xrightarrow{\phi} & H \end{array}$$

Proof. See the proof of [40, Theorem 20.8, §20]. □

We now proceed to define:

Definition 2.2.15. Let G be a Lie group acting on a manifold M, and let $\mathfrak{g}$ be the Lie algebra of G. For any $x \in \mathfrak{g}$, the vector field v_x defined at $m \in M$ by

$$v_x|_m := \frac{d}{dt}\bigg|_{t=0} (exp(-tx)m)$$

is called the *fundamental vector field* associated to $x \in \mathfrak{g}$.

Example 2.2.16. Let G act on $\mathfrak{g}$ by the adjoint action. At any $y \in \mathfrak{g}$, the fundamental vector field associated to $x \in \mathfrak{g}$ is given by:

$$\begin{aligned} v_x|_y &= \frac{d}{dt}\bigg|_{t=0} Ad_{exp(-tx)}y \\ &= \frac{d}{dt}\bigg|_{t=0} exp(-tad_x y) \\ &= \frac{d}{dt}\bigg|_{t=0} exp(-t[x,y]) \\ &= -[x,y]. \end{aligned}$$

Example 2.2.17. Let G act on $\mathfrak{g}^*$ by coadjoint action. At any $\xi \in \mathfrak{g}$, the fundamental vector field associated to $x \in \mathfrak{g}$ is given by

$$\langle v_x|_\xi, y\rangle = \frac{d}{dt}\bigg|_{t=0} \langle Ad^*_{exp(-tx)}\xi, y\rangle = \frac{d}{dt}\bigg|_{t=0} \langle \xi, Ad_{exp(tx)}y\rangle = \langle \xi, [x,y]\rangle.$$

The next useful proposition relates the fundamental vector field of $x \in \mathfrak{g}$ to that of $Ad_g x \in \mathfrak{g}$ for $g \in G$.

Proposition 2.2.18. *Let $\psi : G \to Diff(M)$ be a smooth action. Then for any $g \in G$ and $x \in \mathfrak{g}$*

$$v_{Ad_g x}|_m = \psi_{g*}(v_x|_{g^{-1}m})$$

Proof. Indeed,

$$\begin{aligned} v_{Ad_g x}|_m &= \frac{d}{dt}\bigg|_{t=0} exp(-tAd_g x)m \\ &= \frac{d}{dt}\bigg|_{t=0} exp(Ad_g(-tx))m \\ &= \frac{d}{dt}\bigg|_{t=0} (gexp(-tx)g^{-1})m \\ &= \psi_{g*}|_{g^{-1}m}\frac{d}{dt}\bigg|_{t=0} exp(-tx)(g^{-1}m) \\ &= \psi_{g*}(v_x|_{g^{-1}m}), \end{aligned}$$

where in the third equality we used the naturality of the exponential map (Proposition 2.2.14), and in the fourth equality we used the chain rule. □

Proposition 2.2.19. *The map* $v\_ : \mathfrak{g} \to \mathfrak{X}(M), x \mapsto v_x$ *is a Lie algebra homomorphism*

Proof. Fix $m \in M$, and consider the map $\phi_m : G \to M$ given by $\phi_m(g) = gm$. This map is smooth, and the negative of its differential at $g = e$ evaluated at $x \in \mathfrak{g}$ is given by

$$\begin{aligned}(-d_e\phi_m)(x) &= (d_e\phi_m)(-x) \\ &= \frac{d}{dt}\bigg|_{t=0} (exp(t(-x))m) \\ &= v_x|_m.\end{aligned}$$

If hence follows that the map $x \mapsto v_x$ is linear.

To see that it is a Lie algebra homomorphism, note that, by the previous proposition, $v_{Ad_{exp(-ty)}x}|_m = \psi_{(exp(-ty))*}(v_x|_{(exp(ty))m})$. Differentiating both sides of this identity, we get

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} v_{Ad_{exp(-ty)}x}|_m &= \frac{d}{dt}\bigg|_{t=0} \psi_{(exp(-ty))*}(v_x|_{(exp(ty))m}) \Leftrightarrow \\ \frac{d}{dt}\bigg|_{t=0} v_{Exp(ad(-ty)x)}|_m &= \frac{d}{dt}\bigg|_{t=0} \psi_{(exp(-ty))*}(v_x|_{(exp(ty))m}) \Leftrightarrow \\ v_{[x,y]}|_m &= \pounds_{-v_y} v_x = [-v_y, v_x]|_m \Leftrightarrow \\ v_{[x,y]}|_m &= [v_x, v_y]|_m,\end{aligned}$$

where we have denoted by $Exp : \mathfrak{gl}(\mathfrak{g}) \to GL(\mathfrak{g})$ the exponential map of $GL(\mathfrak{g})$. On the left-hand side of the second identity we used the naturality of the exponential map applied to the Lie group homomorphism $Ad : G \to GL(\mathfrak{g})$, on the left-hand side of the third identity we used the linearity of v_x, and finally on the right-hand side of the third identity we used the definition of the Lie derivative of vector fields and the fact that $\{exp(ty)\}_{t\in\mathbb{R}}$ is the flow of $-v_y$. □

We also define fundamental vector fields corresponding to elements of $\wedge^k\mathfrak{g}$. This definition will be useful in the next chapters:

Definition 2.2.20. Let G be a Lie group acting on a manifold M, and let $\mathfrak{g}$ be the Lie algebra of G. Let $p = x_1 \wedge x_2 \wedge \cdots \wedge x_k \in \wedge^k\mathfrak{g}$, and let v_{x_i} be the fundamental vector field corresponding to $x_i \in \mathfrak{g}$ for each $1 \leq i \leq k$. The multivector field $v_p \in \Gamma(\Lambda^k TM)$ given by

$$v_p := v_{x_1} \wedge v_{x_2} \wedge \cdots \wedge v_{x_k}$$

is the called *the fundamental multivector field* corresponding to p.

The notion is extended linearly to all $p \in \wedge^k \mathfrak{g}$.

Next we will introduce the concepts of orbits and stabilizers of a given Lie group action.

Definition 2.2.21. Let $\psi : G \to \mathrm{Diff}(M)$ be a smooth action. The set

$$G \cdot m := \{g \cdot m \mid g \in G\}$$

is called the *orbit* of G through $m \in M$.

Proposition 2.2.22. *Let $\psi : G \to$ Diff(M) be a smooth action, $m \in M$. The orbit $G \cdot m$ is an immersed submanifold of M.*

Proof. See the proof of Proposition 3.2 (1) in [62, Lecture 13-14]. □

Definition 2.2.23. Let $\psi : G \to \mathrm{Diff}(M)$ be a smooth action. The set

$$G_m := \{g \in G \mid g \cdot m = m\}$$

is called the *stabilizer* of $m \in G$.

Proposition 2.2.24. *(Proposition 3.2 (2) in [62, Lecture 13-14]) Let $\psi : G \to$ Diff(M) be a smooth action. The stabilizer G_m is a Lie subgroup of G, with Lie algebra*

$$\mathfrak{g}_m := \{x \in \mathfrak{g} \mid v_x|_m = 0\}$$

Proof. First of all note that it follows from the definition that the stabilizer G_m is a subgroup of G. Next, note that G_m is the preimage of the point $m \in M$ under the map $ev_\psi|_m : G \to M, g \mapsto g \cdot m$. Since the action is smooth, this map is smooth, and a preimage of a point is a closed set. Thus, by the closed subgroup theorem, G_m is a Lie subgroup of G. The Lie algebra of G_m is then, by definition,

$$\begin{aligned} \mathfrak{g}_m :&= \{x \in \mathfrak{g} \mid exp(tx) \in G_m\} \\ &= \{x \in \mathfrak{g} \mid exp(tx) \cdot m = m\} \end{aligned}$$

By differentiating at $t = 0$, we get $\mathfrak{g}_m \subset \{x \in \mathfrak{g} \mid v_x|_m = 0\}$. Conversely, suppose $v_x|_m = 0$. Then $c(t) \equiv m$, $t \in \mathbb{R}$ is an integral curve of the vector field v_x through $m \in M$. Thus, $exp(tx) \cdot m = c(t) = m$, i.e., $exp(tx) \in G_m$ for all $t \in \mathbb{R}$, and $x \in \mathfrak{g}_m$.

□

Definition 2.2.25 (Cotangent lift)**.** Let G be a Lie group acting on manifold M. Then there is an action $\psi : G \to \mathrm{Diff}(T^*M)$ of G on T^*M defined in a following way:

$$\psi_g : T^*_m M \to T^*_{gm} M$$

$$\alpha \mapsto \psi_g(\alpha),$$

where $\psi_g(\alpha)$ is defined by

$$\langle \psi_g(\alpha), v \rangle = \langle \alpha, \psi_{g^{-1}*} v \rangle,$$

for $v \in T_{gm}M$.

This action is called the *cotangent lift* of the action of G on M.

Definition 2.2.26. Let G be a Lie group acting on manifolds M and N. A smooth map $f : M \to N$ satisfying for all $g \in G$ and $m \in M$

$$f(g \cdot m) = g \cdot f(m)$$

is called *equivariant* with respect to the action of G or *G-equivariant*.

Example 2.2.27. The vector bundle projection $\pi : T^*M \to M$ is G-equivariant, where the action on T^*M is the one from Definition 2.2.25.

2.2.2 Lie algebra actions

Definition 2.2.28. Let $\mathfrak{g}$ be a Lie algebra, and M a manifold. A homomorphism of Lie algebras

$$\rho : \mathfrak{g} \to \mathfrak{X}(M)$$

$$x \mapsto \rho_x$$

is called an *action* of $\mathfrak{g}$ on M.

The action is *smooth* if the following map is smooth:

$$ev_\rho : \mathfrak{g} \times M \to TM$$

$$(x, m) \mapsto \rho_x(m)$$

A closely related notion is that of a Lie algebra representation on a vector space:

Definition 2.2.29. Let $\mathfrak{g}$ be a Lie algebra, and V a vector space. Let $\mathfrak{gl}(V)$ be the space of endomorphisms of V, i.e., linear maps from V to itself, equipped with the commutator bracket. A homomorphism of Lie algebras

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

is called a *representation* of $\mathfrak{g}$ on V.

We will also call the vector space V a representation of $\mathfrak{g}$.

Example 2.2.30. The *adjoint action* or the *adjoint representation* of the Lie algebra $\mathfrak{g}$ on itself is given by

$$ad : \mathfrak{g} \to End(\mathfrak{g}) = \mathfrak{gl}(\mathfrak{g})$$

$$x \mapsto ad_x,$$

where $ad_x(y) = [x, y]$.

This representation induces a representation on $\wedge^k \mathfrak{g}$ by the following formula:

$$ad_x(x_1 \wedge ... \wedge x_k) = [x, x_1] \wedge x_2 \wedge ... \wedge x_k + x_1 \wedge [x, x_2] \wedge ... \wedge x_k + \cdots x_1 \wedge x_2 \wedge ... \wedge [x, x_k].$$

Example 2.2.31. The coadjoint representation ad^* of $\mathfrak{g}$ on $\mathfrak{g}^*$ is given by

$$\langle ad_x^* \xi, y \rangle = \langle \xi, ad_{-x} y \rangle = -\langle \xi, [x, y] \rangle$$

The following proposition follows from 2.2.19.

Proposition 2.2.32. *Let G be a Lie group acting on manifold M, and let $\mathfrak{g}$ be its Lie algebra. Then the following map defines an action of $\mathfrak{g}$ on M.*

$$\rho : \mathfrak{g} \to \mathfrak{X}(M)$$

$$x \mapsto \rho_x,$$

where

$$\rho_x(m) := \frac{d}{dt}\bigg|_{t=0} (exp(-tX)m).$$

This action is called the infinitesimal *action of $\mathfrak{g}$ on M.*

One can "integrate" the infinitesimal action of $\mathfrak{g}$ to obtain a Lie group action near the identity $e \in G$. Thus, for connected Lie groups we have:

Proposition 2.2.33. *(Proposition 2.4 in [62, Lecture 13-14]) An action of a connected Lie group G on a manifold M is uniquely determined by the infinitesimal action of $\mathfrak{g}$.*

Moreover, the following theorem due to R. Palais holds:

Theorem 2.2.34. *(R. Palais, [48]) Let G be a connected simply-connected Lie group, let $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ be a Lie algebra action such that the vector field $\rho(x)$ is complete for all $x \in \mathfrak{g}$. Then there exists a unique action $\psi : G \to Diff(M)$ such that the infinitesimal action associated to ψ as in Proposition 2.2.32 is ρ.*

Proof. [48] or [62, Thm 2.5, Lecture 13-14]. □

2.2.3 Lie algebra homology and cohomology.

This subsection focuses on Lie algebra homology and cohomology, which will be indispensable throughout this thesis. Lie algebra cohomology was introduced by E. Cartan to study topological properties of compact Lie groups using its Lie algebra ([11]), and since its introduction has proved useful in many areas of mathematics and physics (see, e.g., [12]).

Definition 2.2.35. Let $\mathfrak{g}$ be a Lie algebra. The map $\delta_k : \Lambda^k\mathfrak{g} \to \Lambda^{k-1}\mathfrak{g}$ defined by

$$\delta_k : x_1 \wedge ... \wedge x_k \mapsto \sum_{1 \le i < j \le k} (-1)^{i+j}[x_i, x_j] \wedge x_1 \wedge ...\widehat{x_i} \wedge ... \wedge \widehat{x_j} \wedge ...x_k, \quad (2.1)$$

where $k \ge 1$ and $x_i \in \mathfrak{g}$, is called *k-th Lie algebra homology differential* of $\mathfrak{g}$.

We recall the following definition from [42]:

Definition 2.2.36. The kernel of δ_k is called *the k-th Lie kernel* of $\mathfrak{g}$.

We will denote the k-th Lie kernel of $\mathfrak{g}$ by $P_{k,\mathfrak{g}} := ker\delta_k$ and the direct sum of all the Lie kernels by $P_{\mathfrak{g}} := \bigoplus_{k=0}^{dim\,\mathfrak{g}} P_{k,\mathfrak{g}}$. We will also be interested in the subspace $P_{\ge 1,\mathfrak{g}} := \bigoplus_{k=1}^{dim\,\mathfrak{g}} P_{k,\mathfrak{g}}$.

We state the following useful lemma from [55] without proof.

Lemma 2.2.37. *([55, Lemma 3.12]) Let $\mathfrak{g}$ be a Lie algebra, and let $p \in \wedge^k\mathfrak{g}$, $q \in \wedge^l\mathfrak{g}$. Then*

$$\delta_{k+l}(p \wedge q) = \delta_k(p) \wedge q + (-1)^k p \wedge \delta_l(q) + (-1)^k[p, q], \tag{2.2}$$

where $[x_1 \wedge ... \wedge x_k, y_1 \wedge ... \wedge y_l] = \sum(-1)^{i+j}[x_i, y_j] \wedge x_1 \wedge ... \wedge \hat{x}_i \wedge ... \wedge x_k \wedge y_1 \wedge ... \wedge \hat{y}_j \wedge ... \wedge y_l$.

Proof. See the proof of [55, Lemma 3.12]. □

The following important result follows from the above lemma:

Corollary 2.2.38. *The Lie algebra homology differential differential commutes with the adjoint action, i.e.,*

$$\delta_l(ad_x q) = ad_x \delta_l(q)$$

for any $x \in \mathfrak{g}$, $q \in \wedge^l\mathfrak{g}$.

Proof. By (2.2), we have $\delta_{l+1}(x \wedge q) = -x \wedge \delta_l(q) - ad_x q$, since $\delta_1 = 0$. Then

$$\begin{aligned}\delta_l(ad_x q) &= -\delta_l(\delta_{l+1}(x \wedge q)) - \delta_l(x \wedge \delta_l q)\\ &= -\delta_l(x \wedge \delta_l q)\\ &= x \wedge \delta_{l-1}\delta_l(q) + ad_x \delta_l(q)\\ &= ad_x \delta_l(q),\end{aligned}$$

where in the third equality we used the formula (2.2) again. □

Let $\mathfrak{g}$ be a Lie algebra, and M a representation of $\mathfrak{g}$. Consider the space of linear maps $\wedge^k\mathfrak{g} \to M$:

$$C^k(\mathfrak{g}; M) := Hom(\wedge^k\mathfrak{g}, M) \cong \wedge^k\mathfrak{g}^* \otimes M.$$

Elements of $C^k(\mathfrak{g}; M)$ are called *k-forms on $\mathfrak{g}$ with values in M*, or *k-cochains from $\mathfrak{g}$ to M.* We define a graded differential complex

$$... \xrightarrow{d_{\mathfrak{g}}} C^{k-1}(\mathfrak{g}, M) \xrightarrow{d_{\mathfrak{g}}} C^k(\mathfrak{g}; M) \xrightarrow{d_{\mathfrak{g}}} C^{k+1}(\mathfrak{g}, M) \xrightarrow{d_{\mathfrak{g}}} ...$$

with the differential on $\eta \in C^k(\mathfrak{g}; M)$ given by

$$(d_{\mathfrak{g}}\eta)(x_1, ..., x_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}x_i \cdot \eta(x_1, ..., \hat{x}_i, ..., x_{k+1})$$
$$+ \sum_{i<j}(-1)^{i+j}\eta([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{k+1}),$$

where the dot denotes the representation, and $\hat{x}_i$ denotes omission of that element. The fact that $d_{\mathfrak{g}}$ squares to zero is due to the Jacobi identity and M being a representation of $\mathfrak{g}$.

This complex is called the *Chevalley-Eilenberg complex* ([64, §7.7]) of $\mathfrak{g}$ with values in M, and its cohomology

$$H^k(\mathfrak{g}; M) = \frac{ker\, d_{\mathfrak{g}} : C^k(\mathfrak{g}; M) \to C^{k+1}(\mathfrak{g}; M)}{im\, d_{\mathfrak{g}} : C^{k-1}(\mathfrak{g}; M) \to C^k(\mathfrak{g}; M)}$$

is called the *Lie algebra cohomology* of $\mathfrak{g}$ with values in M.

For $k = 0$ we have

$$H^0(\mathfrak{g}; M) = \{m \in M : x \cdot m = 0 \;\; \forall x \in \mathfrak{g}\},$$

i.e., the zeroth Lie algebra cohomology of $\mathfrak{g}$ with values in M is the space of invariants of M under the action of $\mathfrak{g}$.

2.2.3.1 The case of a trivial module

In the case where $M = \mathbb{R}$, and the action of $\mathfrak{g}$ is trivial, we get

$$(d_{\mathfrak{g}}\eta)(x_1, ..., x_{k+1}) = \sum_{i<j}(-1)^{i+j}\eta([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{k+1}),$$

for $\eta \in C^k(\mathfrak{g}, \mathbb{R}) = \wedge^k\mathfrak{g}^*$. The $d_{\mathfrak{g}}$ defined this way is the dual of δ_k introduced in (2.1)

The differential $d_{\mathfrak{g}}$ has all the information about the Lie bracket of $\mathfrak{g}$, since $d_{\mathfrak{g}} : \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$ is the dual of $\delta_2(x, y) = -[x, y]$, and $d_{\mathfrak{g}}^2 = 0$ is equivalent to the Jacobi identity.

Consider the *commutator ideal* $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ defined by:

$$[\mathfrak{g}, \mathfrak{g}] := \{\text{linear combinations of } [x, y], \; \forall x, y \in \mathfrak{g}\}$$

Then, for $c \in C^1(\mathfrak{g}, \mathbb{R}) = \mathfrak{g}^*$, $d_{\mathfrak{g}} c = 0 \iff c([\mathfrak{g}, \mathfrak{g}]) = 0$, and therefore

$$H^1(\mathfrak{g}, \mathbb{R}) = [\mathfrak{g}, \mathfrak{g}]^0,$$

where $[\mathfrak{g}, \mathfrak{g}]^0 \subset \mathfrak{g}^*$ is the annihilator of $[\mathfrak{g}, \mathfrak{g}]$.

2.2.4 Lie algebra extensions

Let $\mathfrak{g}$ be a Lie algebra.

Definition 2.2.39. Lie algebra $\bar{\mathfrak{g}}$ is called an *extension* of $\mathfrak{g}$ by $\mathfrak{a}$, if there exists a short exact sequence of Lie algebras

$$0 \to \mathfrak{a} \to \bar{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

This extension is called *central* if $\mathfrak{a}$ is in the center of $\bar{\mathfrak{g}}$, i.e., $[\bar{\mathfrak{g}}, \mathfrak{a}]_{\bar{\mathfrak{g}}} = 0$.

Central extensions by $\mathbb{R}$ are equivalently characterized by 2-cocycles $c \in \wedge^2 \mathfrak{g}^*$ in the Lie algebra cohomology of $\mathfrak{g}$ with values in $\mathbb{R}$. Then, $\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ and

$$[(x_1, r_1), (x_2, r_2)]_{\bar{\mathfrak{g}}} = ([x_1, x_2]_{\mathfrak{g}}, c(x_1, x_2)).$$

This bracket satisfies the Jacobi identity because c is a cocycle.

Example 2.2.40. Let $\mathfrak{h}$ be the Heisenberg algebra spanned by

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with the commutator relations $[x, y] = z, [x, z] = 0, [y, z] = 0$. The Heisenberg algebra $\mathfrak{h}$ can be seen as the central extension of the abelian Lie algebra $\mathbb{R}^2$ by $\mathbb{R}$ with the corresponding cocycle given by[4] $c(q, p) = 1, c(q, q) = 0, c(p, p) = 0$, where $p, q \in \mathbb{R}^2$ are the basis of $\mathbb{R}^2$.

Indeed, identifying $x \sim (q, 0)$, $y \sim (p, 0)$, $z \sim (0, 1)$, we get

$$[x, y]_{\mathfrak{h}} = [(q, 0), (p, 0)]_{\mathfrak{h}} = ([q, p]_{\mathbb{R}^2}, c(q, p)) = (0, 1) = z$$

and

$$[x, z]_{\mathfrak{h}} = [(q, 0), (0, 1)]_{\mathfrak{h}} = (0, 0)$$
$$[y, z]_{\mathfrak{h}} = [(p, 0), (0, 1)]_{\mathfrak{h}} = (0, 0).$$

[4] We will later see that this is the canonical symplectic form on $\mathbb{R}^2$.

2.3 Symplectic manifolds

In this section we will introduce symplectic manifolds and provide relevant definitions and examples. The material of this section is mostly based on [8].

2.3.1 Definition and examples.

Definition 2.3.1. A manifold M equipped with a 2-form ω such that

1. $d\omega = 0$
2. at every point $m \in M$, ω_m is nondegenerate, i.e., if $v \in T_mM$ is such that $\omega_m(v, u) = 0 \; \forall u \in T_mM$, then $v = 0$

is called a *symplectic* manifold. The form ω is called a *symplectic form.*

Remark 2.3.2. It follows from the nondegeneracy condition that symplectic manifolds are always even-dimensional.

Example 2.3.3. Any orientable 2-dimensional manifold equipped with an area form.

Example 2.3.4. The manifold $M = \mathbb{R}^{2n}$ with coordinates $q_1, ..., q_n, p_1, ..., p_n$ equipped with a 2-form given by $\omega := \sum_{i=1}^{n} dq_i \wedge dp_i$ is a symplectic manifold.

The following theorem by G. Darboux establishes that locally any symplectic manifold is $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dq_i \wedge dp_i)$:

Theorem 2.3.5. *([8])(G.Darboux) Let (M, ω) be a symplectic manifold. Then for any $p \in M$, there exists a coordinate system $(U, q_1, ..., q_n, p_1, ..., p_n)$ centered at $p \in U \subset M$ such that on U*

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

Proof. See the proof of Theorem 8.1 in [8]. □

Example 2.3.6. Phase space. The following example generalizes the previous example in the case of $Q = \mathbb{R}^n$.

Consider the cotangent bundle T^*Q of a manifold Q, with the projection map

$$\pi : T^*Q \to Q.$$

There is a canonical 1-form θ on T^*Q called the *tautological 1-form* and defined by

$$\theta_\eta(v) := \eta(\pi_* v),$$

for $\eta \in T^*Q$ and $v \in T_\eta T^*Q$. Then $\omega := -d\theta$ is the *canonical symplectic form* on T^*Q. Indeed, it is obvious that this form is closed. To see the non-degeneracy, we will write down ω in coordinates. Let $q = (q_1, ..., q_n)$ be local coordinates on Q, and $p = (p_1, ..., p_n)$ the corresponding coordinates on the fibers. Then

$$\theta = p_i dq_i,$$

and

$$\omega = -\sum_{i=1}^{n} dp_i \wedge dq_i = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

It is clear that this form is non-degenerate, and hence (T^*Q, ω) is a symplectic manifold.

Example 2.3.7. Coadjoint orbits. Let G be a a Lie group, $\mathfrak{g}$ its Lie algebra, $\xi \in \mathfrak{g}^*$, and σ_ξ an orbit of ξ under the coadjoint action of G on $\mathfrak{g}^*$. For any $\xi \in \mathfrak{g}^*$, define a skew-symmetric bilinear form on $\mathfrak{g}$ by $b_\xi(x, y) = \langle \xi, [x, y] \rangle$. Recall from Example 2.2.17 that $\langle \xi, [x, y] \rangle = \langle v_x|_\xi, y \rangle$, i.e., $b_\xi(x, y) = 0$ for all $y \in \mathfrak{g}$ if and only if $v_x|_\xi = 0$, where v_x is the fundamental vector field corresponding to $x \in \mathfrak{g}$. Thus, $ker(b_\xi) = \mathfrak{g}_\xi$, where $\mathfrak{g}_\xi$ denotes the Lie algebra of the stabilizer G_ξ of $\xi \in \mathfrak{g}$ under the coadjoint action of G. Therefore, b induces a well-defined non-degenerate form on the orbit σ_ξ of ξ:

$$\omega_\xi(v_x, v_y) = \langle \xi, [x, y] \rangle \tag{2.3}$$

To show that this form is symplectic, we need to show $d\omega = 0$. Indeed,

$$\begin{aligned} d\omega(v_x, v_y, v_z) &= v_x \cdot \omega(v_y, v_z) - v_y \cdot \omega(v_x, v_z) + v_z \cdot \omega(v_x, v_y) \\ &\quad - \omega([v_x, v_y], v_z) + \omega([v_x, v_z], v_y) - \omega([v_y, v_z], v_x). \end{aligned}$$

Evaluating at ξ,

$$\begin{aligned} d\omega(v_x, v_y, v_z)|_\xi &= v_x \cdot \langle \xi, [y, z] \rangle - v_y \cdot \langle \xi, [x, z] \rangle + v_z \cdot \langle \xi, [x, y] \rangle \\ &\quad - \langle \xi, [[x, y], z] \rangle + \langle \xi, [[x, z], y] \rangle - \langle \xi, [[y, z], x] \rangle. \end{aligned}$$

By Example 2.2.17, this becomes

$$\begin{aligned} d\omega(v_x, v_y, v_z)|_\xi &= \langle \xi, [x,[y,z]]\rangle - \langle \xi, [y,[x,z]]\rangle + \langle \xi, [z,[x,y]]\rangle \\ &- \langle \xi, [[x,y],z]\rangle + \langle \xi, [[x,z],y]\rangle - \langle \xi, [[y,z],x]\rangle \\ &= 2(\langle \xi, [x,[y,z]]\rangle - \langle \xi, [y,[x,z]]\rangle + \langle \xi, [z,[x,y]]\rangle) \\ &= 0, \end{aligned}$$

by Jacobi identity.

Therefore, (σ_ξ, ω) is a symplectic manifold.

2.3.2 Symplectic and Hamiltonian vector fields

Definition 2.3.8. A vector field $v \in \mathfrak{X}(M)$ on a symplectic manifold (M, ω) is a *symplectic vector field* if

$$\pounds_v \omega = 0.$$

We will denote the set of symplectic vector fields by $\mathfrak{X}_{Sympl}(M)$

Definition 2.3.9. ([8, Def. 1.7]) Let (M, ω) and (M', ω') be two symplectic manifolds, and let $\phi : M \to M'$ be a diffeomorphism. Then ϕ is a *symplectomorphism* if $\phi^*\omega' = \omega$.

It is then clear from this definition and the definition of the Lie derivative, that symplectic vector fields are infinitesimal counterparts of symplectomorphisms from a symplectic manifold to itself.

Lemma 2.3.10. *Let (M, ω) be a symplectic manifold. A vector field $v \in \mathfrak{X}(M)$ is a symplectic vector field iff $\iota_v \omega$ is closed.*

Proof. By Cartan's magic formula,

$$\pounds_v \omega = \iota_v d\omega + d\iota_v \omega.$$

The statement follows from $d\omega = 0$. □

Definition 2.3.11. A vector field $v_f \in \mathfrak{X}(M)$ on a symplectic manifold (M, ω) is the *Hamiltonian vector field* corresponding to $f \in C^\infty(M)$ if

$$df = -\iota_{v_f} \omega,$$

where ι_{v_f} denotes contraction with the vector field v_f.

Remark 2.3.12. Note that, due to the non-degeneracy of ω, the corresponding Hamiltonian vector field is unique for each $f \in C^\infty(M)$.

We will denote the set of Hamiltonian vector fields by $\mathfrak{X}_{Ham}(M)$.

Note that $v \in \mathfrak{X}(M)$ is a Hamiltonian vector field iff $i_v\omega$ is exact. Therefore, all Hamiltonian vector fields are also symplectic, i.e., $\mathfrak{X}_{Ham}(M) \subset \mathfrak{X}_{Sympl}(M)$.

Moreover, since restrictions of the map

$$\bar{\omega} : \mathfrak{X}(M) \to \Omega^1(M)$$

$$v \mapsto \iota_v\omega$$

provide isomorphisms $\bar{\omega}|_{\mathfrak{X}_{Sympl}(M)} : \mathfrak{X}_{Sympl}(M) \to \Omega^1_{closed}(M)$ and $\bar{\omega}|_{\mathfrak{X}_{Ham}(M)} : \mathfrak{X}_{Ham}(M) \to \Omega^1_{exact}(M)$, we obtain the following exact sequence of vector spaces:

$$0 \to \mathfrak{X}_{Ham}(M) \to \mathfrak{X}_{Sympl}(M) \to H^1(M) \to 0, \tag{2.4}$$

where $H^1(M)$ is the first de Rham cohomology group[5] of M.

In particular,

Proposition 2.3.13. *If $H^1(M) = 0$, then every symplectic vector field on M is Hamiltonian.*

Example 2.3.14. ([8, §18.1]) Consider S^2 with the cylindrical coordinates (θ, h). Then $\omega = d\theta \wedge dh$ is a symplectic structure on S^2, and the vector field $-\frac{\partial}{\partial\theta}$ is a Hamiltonian vector field corresponding to the height function h.

Lemma 2.3.15. *The Lie bracket of symplectic vector fields is Hamiltonian, i.e.,*

$$[\mathfrak{X}_{Sympl}, \mathfrak{X}_{Sympl}] \subset \mathfrak{X}_{Ham}(M).$$

Proof. Let $v, u \in \mathfrak{X}_{Sympl}$ be symplectic vector fields. Then, by Cartan calculus

$$\begin{aligned}\iota_{[v,u]}\omega &= \pounds_v\iota_u\omega - \iota_u\pounds_v\omega \\ &= d\iota_v\iota_u\omega + \iota_v d\iota_u\omega \\ &= d\iota_v\iota_u\omega,\end{aligned}$$

where in the last equality we used that $\iota_u\omega$ is closed, since $u \in \mathfrak{X}_{Sympl}(M)$. Thus, we get that $\iota_{[v,u]}\omega$ is exact, and therefore $[v, u] \in \mathfrak{X}_{Ham}(M)$. □

[5] For example, if M is compact, this group is finite-dimensional.

Corollary 2.3.16. *$\mathfrak{X}_{Ham}(M)$ is a Lie algebra ideal of $\mathfrak{X}_{Sympl}(M)$.*

In particular, if $H^1(M)$ is endowed with the trivial Lie bracket, then the exact sequence (2.4) is an exact sequence of Lie algebras.

2.3.3 The Lie algebra of observables

Definition 2.3.17. Let (M, ω) be a symplectic manifold. The *Poisson bracket* of $f, g \in C^\infty(M)$ is defined by

$$\{f, g\} := \omega(v_f, v_g),$$

where v_f, v_g are the Hamiltonian vector fields corresponding to f, g respectively.

Note that it follows immediately from the definition that the Poisson bracket is bilinear and antisymmetric:

$$\{f, g\} = \omega(v_f, v_g) = -\omega(v_g, v_f) = -\{g, f\}.$$

Lemma 2.3.18. *Let $f, g \in C^\infty(M)$ and v_f, v_g be the respective Hamiltonian vector fields. Then*

$$\{f, g\} = \pounds_{v_f} g = -\pounds_{v_g} f$$

Proof. Indeed,

$$\begin{aligned} \omega(v_f, v_g) &= -\omega(v_g, v_f) \\ &= -\iota_{v_f}\iota_{v_g}\omega \\ &= \iota_{v_f} dg \\ &= \pounds_{v_f} g \end{aligned}$$

The equality $\{f, g\} = -\pounds_{v_g} f$ now follows from the antisymmetry of the Poisson bracket. □

Corollary 2.3.19. *Poisson bracket of any $f \in C^\infty(M)$ with a constant vanishes, i.e.,*

$$\{f, c\} = 0$$

for any $c = const, f \in C^\infty(M)$.

Proof. This follows both from the previous lemma and from the definition of the Poisson bracket, since the Hamiltonian vector field corresponding to a constant function is 0. □

Proposition 2.3.20. *$(C^\infty(M), \{\ ,\ \})$, i.e., the set of smooth functions on a symplectic manifold equipped with the Poisson bracket, is a Lie algebra.*

Proof. Since the bilinearity and the antisymmetry are obvious, the only thing left to prove is that the Jacobi identity is satisfied, i.e., that

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0.$$

Let's denote the left-hand side of the above equality by J, i.e.,

$$J := \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\}.$$

It suffices to show that

$$2J = d\omega(v_f, v_g, v_h),$$

where v_f, v_g, v_h are the Hamiltonian vector fields of f, g, h respectively. Then $J = 0$ will follow from the fact that $d\omega = 0$. Indeed,

$$\begin{aligned} d\omega &= v_f \cdot \omega(v_g, v_h) - v_g \cdot \omega(v_f, v_h) + v_h \cdot \omega(v_f, v_g) \\ &\quad - \omega([v_f, v_g], v_h) + \omega([v_f, v_h], v_g) - \omega([v_g, v_h], v_f). \end{aligned}$$

It then follows from Lemma 2.3.18 that

$$\begin{aligned} d\omega &= \{f, \{g, h\}\} - \{g, \{f, h\}\} + \{h, \{f, g\}\} \\ &\quad - \omega([v_f, v_g], v_h) + \omega([v_f, v_h], v_g) - \omega([v_g, v_h], v_f) \\ &= J - \omega([v_f, v_g], v_h) + \omega([v_f, v_h], v_g) - \omega([v_g, v_h], v_f). \end{aligned} \tag{2.5}$$

Using Cartan calculus, invariance of ω, and the definition of Hamiltonian vector fields, we get

$$\begin{aligned} \iota_{[v_f, v_g]}\omega &= \pounds_{v_f}\iota_{v_g}\omega - \iota_{v_g}\pounds_{v_f}\omega \\ &= \pounds_{v_f}\iota_{v_g}\omega \\ &= d\iota_{v_f}\iota_{v_g}\omega + \iota_{v_f}d\iota_{v_g}\omega \\ &= d\iota_{v_f}\iota_{v_g}\omega - \iota_{v_f}ddg \\ &= d\iota_{v_f}\iota_{v_g}\omega \\ &= d\{g, f\}. \end{aligned}$$

Thus,

$$\omega([v_f, v_g], v_h) = \iota_{v_h}d\{g, f\} = \pounds_{v_h}\{g, f\} = \{h, \{g, f\}\},$$

and equation (2.5) becomes

$$\begin{aligned} d\omega &= J - \omega([v_f, v_g], v_h) + \omega([v_f, v_h], v_g) - \omega([v_g, v_h], v_f) \\ &= J - \{h, \{g, f\}\} + \{g, \{h, f\}\} - \{f, \{h, g\}\} \\ &= 2J. \end{aligned}$$

□

Definition 2.3.21. The Lie algebra $(C^\infty(M), \{\ ,\ \})$ is called the *Lie algebra of observables.*[6]

Proposition 2.3.22. *If v_f, v_g are Hamiltonian vector fields corresponding respectively to $f, g \in C^\infty(M)$, then $[v_f, v_g]$ is the Hamiltonian vector field corresponding to $\{f, g\}$. In particular, the map $f \mapsto v_f$, assigning Hamiltonian vector fields to smooth functions on M, is a Lie algebra morphism.*

Proof. By Cartan calculus,

$$\begin{aligned} \iota_{[v_f, v_g]}\omega &= \pounds_{v_f}\iota_{v_g}\omega - \iota_{v_g}\pounds_{v_f}\omega \\ &= d\iota_{v_f}\iota_{v_g}\omega + \iota_{v_f}d\iota_{v_g}\omega \\ &= d\iota_{v_f}\iota_{v_g}\omega \\ &= -d\{f, g\}. \end{aligned}$$

□

2.4 Moment maps in symplectic geometry

2.4.1 History and motivation

The notion of moment map is extremely important in symplectic geometry. It is instrumental in formulating a Hamiltonian version of the celebrated Noether theorem, plays an essential role in construction of symplectic quotients, and in the characterization of torus actions on symplectic manifolds. The name comes from the fact that moment maps generalize the classical notions of linear and angular momentum, as will be demonstrated in the examples below.

[6] The name comes from physics, where an *observable* is a quantity that can be measured in principle. In particular, in classical mechanics *observables* are smooth functions on the phase space of a system, which is a symplectic manifold.

Moment maps had appeared in classical mechanics before they were defined in general by Kostant and Souriau in the late 60's ([25]). They generalize to arbitrary Lie groups the classical notions of total linear or angular momentum ([23]) and "enable one to relate the geometry of a symplectic manifold with symmetry to the structure of its symmetry group" [66].

Moment maps have numerous applications in mathematics and physics. One of the most fundamental applications of moment maps is the assignment of conserved quantities to symmetries of a physical system in accordance with Noether's first theorem (see, e.g., [8, §24]; an excellent source on the Lagrangian formulation and the history and influence of Noether's theorems is [38]):

Theorem 2.4.1. *(E. Noether) Consider the physical system given by* (M, ω, H)*, where* (M, ω) *is a symplectic manifold, and* $H \in C^\infty(M)$ *is the Hamiltonian. Let* G *be a connected Lie group acting on* (M, ω)*, and let* $\mu : M \to \mathfrak{g}^*$ *be a moment map for this action. Then* H *is* G*-invariant if and only if* μ *is constant along the flow of* v_H*.*

This is directly related to another important application of the moment map - the "symplectic reduction", i.e., the process of reducing the number of variables describing the physical system by exploiting symmetries and conserved quantities (see [8], [45], [47]):

Theorem 2.4.2. *(J. Marsden - A. Weinstein, K. Meyer) Let* G *be a Lie group acting on a symplectic manifold* (M, ω)*, and let* $\mu : M \to \mathfrak{g}^*$ *be a moment map for this action. Assume that* G *acts freely and properly* [7] *on* $\mu^{-1}(0)$*, and let* $i : \mu^{-1}(0) \to M$ *be the inclusion map. Then*

- *the orbit space* $M_{red} := \mu^{-1}(0)/G$ *is a manifold*
- $\pi : \mu^{-1} \to M_{red}$ *is a principal* G*-bundle*
- *there is a unique symplectic form* ω_{red} *on* M_{red}*, such that* $i^*\omega = \pi^*\omega_{red}$*.*

Moment maps also play an important role in the classification of symplectic toric manifolds, i.e., compact connected symplectic manifolds (M, ω) of dimension $2n$ equipped with effective actions of a torus $\mathbb{T}^n$ together with moment maps $\mu : M \to Lie(\mathbb{T}^n) = \mathbb{R}^n$. Before stating the classification result, we need the following definition, taken almost verbatim from [8, §28].

Definition 2.4.3. A *Delzant polytope* in $\mathbb{R}^n$ is a convex polytope that

- is *simple*, i.e., there are n edges meeting at each vertex

[7] It can be shown that 0 is a regular value of the moment map.

- is *rational*, i.e., all edges meeting at the vertex p are of the form $p + tu_i$, for $t \geq 0$, $u_i \in \mathbb{Z}^n$
- is smooth, i.e., for each vertex, the corresponding u_i can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

We can now state the theorem: ([8, §28], [15]):

Theorem 2.4.4. *There is a one-to-one correspondence between symplectic toric manifolds (M, ω) with moment maps $\mu : M \to \mathbb{R}^n$ and Delzant polytopes:*

$$\{toric\ \ manifolds\} \to \{Delzant\ \ polytopes\}$$

$$(M, \omega) \mapsto \mu(M)$$

As the final example of well-known applications of moment maps we will briefly mention the "orbit method" by A. Kirillov: It says that a symplectic manifold equipped with a transitive action of a Lie group G that admits a moment map corresponds to an irreducible unitary representation of G. Moreover, every such symplectic manifold corresponds to a coadjoint orbit of G on $\mathfrak{g}^*$. Thus, irreducible representations of Lie groups correspond to coadjoint orbits of these groups. In particular, for simply connected nilpotent Lie groups this method allows to explicitly construct a bijection between coadjoint orbits and irreducible unitary representations (see [35] and [61] for a short summary).

We end this introduction by mentioning the link between moment maps and equivariant de Rham cohomology. Namely, let G be a compact Lie group acting on a manifold M. Consider the space

$$C_G(M) := (S(\mathfrak{g}^*) \otimes \Omega(M))^G,$$

i.e., the G-invariant elements of $S(\mathfrak{g}^*) \otimes \Omega(M)$, where $S(\mathfrak{g}^*)$ is the symmetric algebra on $\mathfrak{g}^*$. The Lie group G acts by coadjoint representation on $S(\mathfrak{g}^*)$, and by pull-back on $\Omega(M) : g \cdot \omega := (g^{-1})^* \omega$. Note that $S(\mathfrak{g}^*) \otimes \Omega(M)$ is a graded vector space, where elements of $\mathfrak{g}^*$ are considered to have degree 2, and the degree of $\eta \in \Omega^k(M)$ is k, i.e., the usual degree of differential forms.

Consider a degree 2-element $(\eta + f) \in (S(\mathfrak{g}^*) \otimes \Omega(M))^G$, where $\eta \in \Omega^2(M)$, and $f \in \mathfrak{g}^* \otimes C^\infty(M)$. We can define the *equivariant differential* d_G on such elements by

$$d_G(\eta + f)(x) := d\eta + \iota_{v_x}\eta + d(f(x)),$$

where $v_x \in \mathfrak{X}(M)$ is the fundamental vector field corresponding to $x \in \mathfrak{g}$.

Now consider a compact connected Lie group G acting on a symplectic manifold (M, ω) by preserving ω, and an element $\mu^* \in (\mathfrak{g} \otimes C^\infty(M))^G$, which can be seen as an equivariant map $\mathfrak{g} \to C^\infty(M)$. Consider a degree 2-element $(\omega + \mu^*) \in (S(\mathfrak{g}^*) \otimes \Omega(M))^G$. Then

$$d_G(\omega + \mu^*)(x) = d\omega + \iota_{v_x}\omega + d(\mu^*(x))$$
$$= \iota_{v_x}\omega + d(\mu^*(x)).$$

Thus, $d_G(\omega + \mu^*) = 0$ if and only $\iota_{v_x}\omega + d(\mu^*(x)) = 0$, i.e., μ^* is a comoment map, and $\mu : M \to \mathfrak{g}^*, \mu(m)(y) = \mu^*(y)(m)$ for all $y \in \mathfrak{g}$, is a moment map.

For a thorough treatment of this topic see [1] and [19].

2.4.2 Definition and examples

It is common in mathematics to consider symmetries of various structures, i.e., Lie group actions that preserve the given structure. It is thus natural to consider actions of Lie groups on symplectic manifolds that preserve the symplectic structure. Then Lemma 2.3.10 tells us that, for fundamental vector fields v_x of the action corresponding to $x \in \mathfrak{g}$, the 1-form $\iota_{v_x}\omega$ will be closed. We can take a step further and require these forms to be exact, and thus the fundamental generators of the action to be Hamiltonian: by (2.4), the quotient $\mathfrak{X}_{Sympl}(M)/\mathfrak{X}_{Ham}(M) = H^1(M)$, which is often finite-dimensional, e.g., for compact manifolds. This leads to the following definition:

Definition 2.4.5. Let G be a Lie group acting on a symplectic manifold (M, ω). This action is a *Hamiltonian action* if there exists a map

$$\mu : M \to \mathfrak{g}^*$$

satisfying the following properties:

1. For each $x \in \mathfrak{g}$, let $\mu^x : M \to \mathbb{R}$ be defined as[8] $\mu^x(m) := \mu(m)(x)$. Let v_x be the fundamental vector field associated to x. Then

$$d\mu^x = -\iota_{v_x}\omega.$$

2. μ is equivariant with respect to the given action of G on M and the coadjoint action of G on $\mathfrak{g}^*$, i.e., $\forall g \in G, m \in M$,

$$\mu(gm) = Ad_g^*(\mu(m))$$

[8] This is the x-component of the moment map, i.e., the composition of μ with the linear map $x\colon \mathfrak{g}^* \to \mathbb{R}$

Then (M, ω, G, μ) is called a *Hamiltonian G-space*, and μ is called a *moment map*.

Definition 2.4.6. Let G be a Lie group acting on a symplectic manifold (M, ω). A *comoment map* for this action is a linear map

$$\mu^* : \mathfrak{g} \to C^\infty(M)$$

satisfying the following properties:

1. For each $x \in \mathfrak{g}$ let v_x the fundamental vector field corresponding to $x \in \mathfrak{g}$. Then
$$d(\mu^*(x)) = -\iota_{v_x}\omega.$$
2. $\mu^*(x)$ is a Lie algebra morphism:
$$\mu^*([x, y]) = \{\mu^*(x), \mu^*(y)\},$$
where $\{\ ,\ \}$ is the Poisson bracket on $C^\infty(M)$.

Thus, a comoment map is a lift of the Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}_{Ham}(M)$:

$$\begin{array}{ccc} & & C^\infty(M) \\ & \nearrow^{\mu^*} & \downarrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{X}_{Ham}(M) \end{array}$$

where the horizontal map is the Lie algebra action, and the vertical map assigns to a function on M its Hamiltonian vector field.

The next proposition says that for connected Lie groups the two previous definitions are equivalent:

Proposition 2.4.7. *Let G be a connected Lie group acting on a symplectic manifold (M, ω). A map $\mu : M \to \mathfrak{g}^*$ is a moment map if and only if the "dual" map $\mu^* : \mathfrak{g} \to C^\infty(M)$ defined by $\mu^*(x)(m) := \mu(m)(x)$ is a comoment map.*

Proof. This proof follows very closely that of [63, Prop. 1.2, Lecture 8].

The statement of the proposition is clear regarding the condition $d\mu^*(x) = -\iota_{v_x}\omega$. Thus, it remains to prove that $\mu : M \to \mathfrak{g}^*$ is equivariant if and only if $\mu^* : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra morphism.

Assume $\mu : M \to \mathfrak{g}^*$ is equivariant. Then

$$
\begin{aligned}
\{\mu^*(x), \mu^*(y)\}|_m &= \pounds_{v_x}\mu^*(y)|_m \\
&= \frac{d}{dt}\bigg|_{t=0} \mu^*(y)(exp(-tx)m) \\
&= \frac{d}{dt}\bigg|_{t=0} \langle \mu(exp(-tx)m), y\rangle \\
&= \frac{d}{dt}\bigg|_{t=0} \langle Ad^*_{exp(-tx)}\mu(m), y\rangle \quad \text{using the equivariance assumption} \\
&= \frac{d}{dt}\bigg|_{t=0} \langle \mu(m), Ad_{exp(tx)}y\rangle \\
&= \frac{d}{dt}\bigg|_{t=0} \langle \mu(m), Exp(ad(tx))y\rangle \quad \text{using the naturality of } exp \\
&= \langle \mu(m), [x, y]\rangle \\
&= \mu^*([x, y])|_m.
\end{aligned}
$$

Thus, if μ is equivariant, μ^* is a Lie algebra homomorphism.

Conversely, suppose $\mu^* : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra homomorphism. We wish to prove that $\mu : M \to \mathfrak{g}^*$ is equivariant.

It is known from theory of Lie groups (see, e.g., [40]) that:

1. $exp : \mathfrak{g} \to G$ is a diffeomorphism from a neighborhood of $0 \in T_eG$ to a neighborhood of $e \in G$
2. a connected Lie group G is generated by any neighborhood of the identity $e \in G$.

Taking into account these two facts, it will suffice to prove that

$$
\mu(exp(tx)m) = Ad^*_{exp(tx)}\mu(m) \tag{2.6}
$$

for all $x \in \mathfrak{g}$.

Recall that, if two vector fields v and u are f-related, i.e., $(df)(v|_m) = u_{f(m)}$ for all $m \in M$, then we have the following relationship between their flows ϕ^v_t

and ϕ_t^u:

$$f(\phi_t^v(m)) = \phi_t^u(f(m)).$$

Let $v_x^{Ad^*}$ denote the fundamental vector field for the coadjoint action corresponding to $x \in \mathfrak{g}$. It then follows that, if we can prove that v_x and $v_x^{Ad^*}$ are μ-related, i.e.,

$$(d\mu)(v_x|_m) = v_x^{Ad^*}|_{\mu(m)} \tag{2.7}$$

for all $x \in \mathfrak{g}$, we can conclude

$$\mu(exp(-tx)m) = Ad^*_{exp(-tx)}\mu(m)$$

for all $x \in \mathfrak{g}$, which is equivalent to (2.6).

Indeed, we have for all $y \in \mathfrak{g} \simeq (\mathfrak{g}^*)^*$:

$$\begin{aligned}
\langle (d\mu)(v_x|_m), y\rangle &= y((d\mu)(v_x|_m)) \\
&= d(y \circ \mu)(v_x|_m) \quad \text{using that } y \text{ is a linear function on } \mathfrak{g}^* \\
&= \mathcal{L}_{v_x}(y \circ \mu)|_m \\
&= \mathcal{L}_{v_x}(\mu^*(y))|_m \\
&= \{\mu^*(x), \mu^*(y)\}|_m \\
&= \mu^*([x,y])|_m \quad \text{using the assumption that } \mu^* \text{ is a Lie algebra morphism} \\
&= \langle \mu(m), [x,y]\rangle \\
&= \langle v_x^{Ad^*}|_{\mu(m)}, y\rangle \quad \text{using Example 2.2.17}
\end{aligned}$$

Thus, $\langle (d\mu)(v_x|_m), y\rangle = \langle v_x^{Ad^*}|_{\mu(m)}, y\rangle$ for all $y \in \mathfrak{g}$, i.e., we have proved formula (2.7).

□

Sometimes in the literature maps only satisfying the first requirement of the above definitions are considered. The action of G on (M, ω) possessing such a map is then called *weakly Hamiltonian.* This is the motivation to introduce the following definition:

Definition 2.4.8. ([29]) Let G be a Lie group acting on a symplectic manifold (M, ω). A map

$$\mu : M \to \mathfrak{g}^*$$

is called a *weak moment map* for this action if

$$d(\mu^*(x)) = -\iota_{v_x}\omega,$$

for each $x \in \mathfrak{g}$ and the dual map $\mu^* : \mathfrak{g} \to C^\infty(M)$ defined by

$$\mu^*(x)(m) := \mu(m)(x).$$

The dual map $\mu^* : \mathfrak{g} \to C^\infty(M)$ is called a *weak comoment map.*

We note that the second condition in the definition of the moment map, i.e., the equivariance condition, is required for many applications listed in §2.4.1, such as symplectic reduction, classification of toric manifolds, and the orbit method. If we have a weak moment map, the equivariance can often be achieved by "averaging" over a compact Lie group or a compact manifold: e.g., if the Lie group G is compact or the manifold M is compact and equipped with a G-invariant volume form.

We now present examples of moment maps.

The following two examples from classical mechanics are the reason behind the name "moment map".

Example 2.4.9. Linear momentum. Consider $\mathbb{R}^6$ with coordinates $(q, p) = ((q_1, q_2, q_3), (p_1, p_2, p_3))$ and symplectic form $\omega := \sum_{i=1}^{3} dq_i \wedge dp_i$. Let $G = \mathbb{R}^3$ act on $(\mathbb{R}^6, \omega)$ by translations, i.e., for $g \in \mathbb{R}^3$ and $(q, p) \in \mathbb{R}^6$,

$$g(q, p) := (q + g, p).$$

This action leaves ω invariant, and for $x = (x_1, x_2, x_3) \in \mathbb{R}^3 = \mathfrak{g}$, the fundamental vector field is $v_x = -x_1 \frac{\partial}{\partial q_1} - x_2 \frac{\partial}{\partial q_2} - x_3 \frac{\partial}{\partial q_3}$.

Then

$$\mu : \mathbb{R}^6 \to \mathbb{R}^3$$

$$(q, p) \mapsto p$$

is a moment map, and

$$\begin{aligned} \mu^*(x)|_{(q,p)} &= \langle \mu(q,p), x \rangle \\ &= \vec{p} \cdot \vec{x} \end{aligned}$$

is a comoment map. In classical mechanics, the vector $\vec{p}$ is called the *momentum vector* corresponding to the *position vector* $\vec{q}$, and the map μ is called the *linear momentum.*

Example 2.4.10. Angular momentum. As in the example above, consider $\mathbb{R}^6$ with coordinates $(q,p) = ((q_1,q_2,q_3),(p_1,p_2,p_3))$ and symplectic form $\omega := \sum_{i=1}^{3} dq_i \wedge dp_i$. Let $G = SO(3)$ act on $\mathbb{R}^3$ by matrix multiplication, $g(q) := gq$ for any $g \in G$. The induced action on T^*Q (see Definition 2.2.25) is given by

$$\begin{aligned} g(q,p) &= (gq, p \circ g^{-1}_{gq*}) \\ &= (gq, p \circ g^{-1}) \\ &= (gq, pg^T), \end{aligned}$$

since $g^T g = Id$, $\forall g \in SO(3)$.

Then, for $x \in \mathfrak{g} = \mathfrak{so}(3)$ and $(q,p) \in T^*Q$,

$$\begin{aligned} v_x(q,p) &= \left.\frac{d}{dt}\right|_{t=0} (exp(-tx)(q,p)) \\ &= \left.\frac{d}{dt}\right|_{t=0} (exp(-tx)q, pexp(-tx^T)) \\ &= (-xq, px), \end{aligned}$$

since $-x^T = x$,$\forall x \in \mathfrak{so}(3)$.

Recall that $\mathfrak{so}(3)$ can be identified with $\mathbb{R}^3$ equipped with the cross product of vectors, under the following map:

$$I : \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Under this identification, $xq = I(x) \times q$ and $px = p^T \times I(x)$, and one can show that

$$\mu : \mathbb{R}^6 \to \mathbb{R}^3$$
$$(q,p) \mapsto q \times p$$

is a moment map, and

$$\begin{aligned}\mu^*(x)|_{(q,p)} &= \langle \mu(q,p), x\rangle \\ &= (q \times p)\cdot I(x)\end{aligned}$$

is a comoment map. In classical mechanics, the map μ is called the *angular momentum*.

Example 2.4.11. Phase space. The above two examples are special cases of the following example. Let Q be a manifold, and consider $(M = T^*Q, \omega)$ as in Example 2.3.6. Let G be a Lie group acting on Q, and consider the lifted action ψ of G on T^*Q. Then ω is invariant under this action. Indeed, let θ denote the tautological 1-form on T^*Q. To make the notation less busy, we will denote the diffeomorphism ψ_g by just g for $g \in G$. Then for $\eta \in T^*Q$ and $u \in T_\eta(T^*Q)$,

$$\begin{aligned}(g^*\theta)_\eta(u) &= \theta_{g\eta}(g_{\eta *}u) \\ &= (g\eta)(\pi_{g\eta *}(g_{\eta *}(u))) \\ &= (g\eta)((\pi \circ g)_{\eta *}(u)) \\ &= \eta(g^{-1}_{\pi g\eta *}((\pi \circ g)_{\eta *}(u))) \\ &= \eta((g^{-1} \circ \pi \circ g)_{\eta *}(u)) \\ &= \eta(\pi_{\eta_*}(u)),\end{aligned}$$

where in the last equality we have used that $\pi : T^*Q \to Q$ is G-equivariant. Therefore, $\pounds_{v_x}\theta = 0\ \forall x \in \mathfrak{g}$, and hence, since the Lie derivative commutes with d, we have

$$\pounds_{v_x}\omega = -\pounds_{v_x}d\theta = -d\pounds_{v_x}\theta = 0$$

for all $x \in \mathfrak{g}$. Also note that, since $0 = \pounds_{v_x}\theta = d\iota_{v_x}\theta + \iota_{v_x}d\theta$,

$$d\iota_{v_x}\theta = -\iota_{v_x}d\theta = \iota_{v_x}\omega$$

Therefore,

$$\mu^* : \mathfrak{g} \to C^\infty(M)$$
$$x \mapsto -\iota_{v_x}\theta$$

is a weak co-moment map.

The weak moment map is then given by

$$\mu : M \to \mathfrak{g}^*, \mu(m)(x) = -(\iota_{v_x}\theta)(m).$$

To show equivariance, we have to show

$$\mu(g \cdot m)(x) = Ad_g(\mu(m))(x) = \mu(m)(Ad_{g^{-1}}(x))$$

i.e.,

$$(\iota_{v_x}\theta)(g \cdot m) = (\iota_{v_{Ad_{g^{-1}}x}}\theta)(m).$$

Looking closer at the left-hand side, we obtain

$$\begin{aligned}\theta_{gm}(v_x|_{gm}) &= \theta_{gm}(\psi_{g*}\psi_{g^{-1}*}(v_x|_{gm}))\\ &= (\psi_g^*\theta)_m(\psi_{g^{-1}*}(v_x|_{gm}))\\ &= \theta_m(\psi_{g^{-1}*}(v_x|_{gm}))\\ &= \theta_m(v_{Ad_{g^{-1}}x}|_m),\end{aligned}$$

where in the second-to-last equality we used the G-invariance of θ, and in the last equality we used Proposition 2.2.18.

Example 2.4.12. Exact symplectic forms.

The previous example can be straightforwardly generalized to the action of a Lie group G on a general symplectic manifold (M, ω) where $\omega = -d\theta$ for a G-invariant θ. A moment map for the action of G on (M, ω) is given by $\mu : M \to \mathfrak{g}^*, \mu(m)(x) = -(\iota_{v_x}\theta)(m)$.

Example 2.4.13. Coadjoint orbits. Let G be a a Lie group, $\mathfrak{g}$ its Lie algebra, $\xi \in \mathfrak{g}^*$, and σ_ξ an orbit of ξ under the coadjoint action of G on $\mathfrak{g}^*$. Consider (σ_ξ, ω) as in Example 2.3.7.

Note that ω is G-invariant. Indeed, for any $g \in G$

$$
\begin{aligned}
(\psi_g^*\omega)_\xi(v_x, v_y) &= \omega_{g\xi}(\psi_{g*}(v_x|_\xi), \psi_{g*}(v_y|_\xi)) \\
&= \omega_{g\xi}(v_{Ad_g x}|_{g\xi}, v_{Ad_g y}|_{g\xi}) \\
&= \langle g\xi, [Ad_g x, Ad_g y]\rangle \\
&= \langle g\xi, Ad_g[x, y]\rangle \\
&= \langle \xi, Ad_{g^{-1}} Ad_g[x, y]\rangle \\
&= \langle \xi, [x, y]\rangle \\
&= \omega_\xi(v_x, v_y),
\end{aligned}
$$

where in the second equality we used Proposition 2.2.18, and in the fourth equality we used that $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism.

The moment map $\mu : \sigma_\xi \to \mathfrak{g}^*$ for the action of G on (σ_ξ, ω) is given by inclusion $\sigma_\xi \to \mathfrak{g}^*$. Indeed, evaluating on fundamental vector fields of the action, which form the tangent space to the orbit at any point, we get:

$$
\begin{aligned}
(d\mu^*(x))(v_y)|_\xi &= v_y(\mu(\xi)(x)) \\
&= v_y\langle \xi, x\rangle \\
&= \frac{d}{dt}|_{t=0}(\langle Ad_{exp(-ty)}\xi, x\rangle) \\
&= \frac{d}{dt}|_{t=0}(\langle \xi, Ad_{exp(ty)}x\rangle) \\
&= \langle \xi, [y, x]\rangle \\
&= -\langle \xi, [x, y]\rangle \\
&= -\omega_\xi(v_x, v_y),
\end{aligned}
$$

i.e., $d\mu^*(x) = -\iota_{v_x}\omega$. Note that this moment map is clearly equivariant.

2.4.3 Existence and uniqueness

This subsection addresses the questions of existence and uniqueness of moment maps. The content of this section is based on [8, §26] and [23, §24], as well as on [2], [31], [46], and [57]

2.4.3.1 Uniqueness.

Let's start from the simpler question: given a moment map for an action of G on (M, ω), how many more are there?

Proposition 2.4.14. *Let G be a Lie group acting on a connected symplectic manifold (M, ω). If μ and μ' are two moment maps for this same action, then*

$$\mu - \mu' \in [\mathfrak{g}, \mathfrak{g}]^0 = H^1(\mathfrak{g}, \mathbb{R})$$

Proof. Let μ and μ' be two moment maps for the action of G on (M, ω). Then, for all $x \in \mathfrak{g}$,

$$d(\mu^*(x) - \mu'^*(x)) = 0,$$

and $b^x := \mu^*(x) - \mu'^*(x)$ is a locally constant function on M. By assumption, M is connected. Then, since b^x is linear in x, it defines an element $b \in \mathfrak{g}^*$ by

$$\langle b, x \rangle = b^x, \tag{2.8}$$

for any $x \in \mathfrak{g}$.

Then $\mu = \mu' + b$, i.e., two moment maps for the same Lie group action differ by an element of $\mathfrak{g}^*$. Further, since $\mu^*(x)$ and $\mu'^*(x)$ are Lie algebra morphisms, $\forall x, y \in \mathfrak{g}$ we have

$$\begin{aligned} b^{[x,y]} &= \mu^*([x, y]) - \mu'^*([x, y]) \\ &= \{\mu^*(x), \mu^*(y)\} - \{\mu'^*(x), \mu'^*(y)\} \\ &= \{\mu'^*(x) + b^x, \mu'^*(y) + b^y\} - \{\mu'^*(x), \mu'^*(y)\} \\ &= 0, \end{aligned}$$

since Poisson bracket of any function with a constant vanishes. Thus, the element $b \in \mathfrak{g}^*$ defined in (2.8) satisfies

$$\langle b, [\mathfrak{g}, \mathfrak{g}] \rangle = 0,$$

i.e., $b \in [\mathfrak{g}, \mathfrak{g}]^0 = H^1(\mathfrak{g}, \mathbb{R})$.

□

Note that it follows from the proof above that:

Corollary 2.4.15. *If $\mu : M \to \mathfrak{g}^*$ is a moment map, then $\mu' := \mu + b$ for $b \in H^1(\mathfrak{g}, \mathbb{R})$ is another moment map for the same action.*

2.4.3.2 Existence.

Let G be a Lie group acting on a connected symplectic manifold (M, ω) by symplectomorphisms. When does this action admit a moment map?[9]

Consider the map $h : C^\infty(M) \to \mathfrak{X}_{Ham}(M)$, assigning to a function in $C^\infty(M)$ its Hamiltonian vector field. By Proposition 2.3.22, this map is a Lie algebra morphism, and its kernel consists of constant functions on M, i.e., we have the following short exact sequence of Lie algebras

$$0 \to \mathbb{R} \to C^\infty(M) \xrightarrow{h} \mathfrak{X}_{Ham}(M) \to 0.$$

Thus the Poisson algebra $(C^\infty(M), \{\ ,\ \})$ is a central extension of $(\mathfrak{X}_{Ham}(M), [\ ,\ ])$ by $(\mathbb{R}, [\ ,\ ] = 0)$.

Combining the exact sequence above with the sequence (2.4), we obtain the following exact sequence.

$$0 \to \mathbb{R} \to C^\infty(M) \to \mathfrak{X}_{Sympl}(M) \to H^1(M) \to 0. \tag{2.9}$$

Consider the Lie algebra action map $\rho : \mathfrak{g} \to \mathfrak{X}_{Sympl}(M)$. In order for a moment map for this action to exist, the map ρ should admit a lift to $C^\infty(M)$ that is also a Lie algebra morphism. Such a linear lift exists if and only if the image of $\rho(\mathfrak{g})$ lies in $\mathfrak{X}_{Ham}(M)$. In other words, the map $\mathfrak{g} \to \mathfrak{X}_{Sympl}(M) \to H^1(M)$, $x \mapsto [\iota_{v_x}\omega]$ has to vanish. By Lemma 2.3.15, this map defines a 1-cocycle in the Lie algebra cohomology $H^1(\mathfrak{g}, H^1(M))$ of $\mathfrak{g}$ with values in the trivial module $H^1(M)$. Its class is the obstruction to lifting the map $\rho : \mathfrak{g} \to \mathfrak{X}_{Sympl}(M)$ to a linear map $C^\infty(M)$. Since $H^1(\mathfrak{g}, H^1(M)) = H^1(\mathfrak{g}) \otimes H^1(M)$, we have the following corollary:

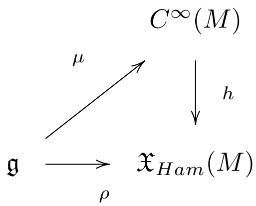
Corollary 2.4.16. *Let G act on (M, ω) by symplectomorphisms. If $H^1(\mathfrak{g}) = 0$ or $H^1(M) = 0$, then this action is weakly Hamiltonian.*

Example 2.4.17. Consider S^1 acting on a 2-torus $(\mathbb{T} = S^1 \times S^1, \omega := dt_1 \wedge dt_2)$ via $\theta \cdot (t_1, t_2) := (\theta + t_1, t_2)$, where t_i are the periodic (angular) coordinates. Note that $H^1(\mathfrak{g}) = H^1(\mathbb{R}) = \mathbb{R} \neq 0$ and $H^1(\mathbb{T}) = \mathbb{R}^2 \neq 0$.

The infinitesimal generator of the action corresponding to $1 \in \mathbb{R} = \mathfrak{g}$ is $v = \frac{\partial}{\partial t_1}$. Since $-\iota_v\omega = dt_2$, and the form dt_2 is closed, $v = \frac{\partial}{\partial t_1}$ is symplectic. However, it is not Hamiltonian, since dt_2 is not exact: note that dt_2 is not an exterior derivative of the angular coordinate t_2, because the angular coordinate is not a well-defined function on all of $\mathbb{T}^2$.

[9] The exposition below was influenced by Jose Figueroa-O'Farrill's post in the following thread: https://mathoverflow.net/questions/55442/why-can-we-define-the-moment-map-in-this-way-i-e-why-is-this-form-exact

Now assume that we have a *linear* map $\mu : \mathfrak{g} \to C^\infty(M)$ that lifts $\rho : \mathfrak{g} \to \mathfrak{X}_{Ham}(M)$, i.e., $h \circ \mu = \rho$.[10]



Since ρ is a Lie algebra homomorphism, for $x, y \in \mathfrak{g}$ we have

$$h(\mu([x,y]) - \{\mu(x), \mu(y)\}) = 0.$$

Hence, $\mu([x,y]) - \{\mu(x), \mu(y)\} \in ker(h) = \mathbb{R}$ for all $x, y \in \mathfrak{g}$, and thus we can define $c \in \wedge^2 \mathfrak{g}^*$ by

$$c(x,y) = \mu([x,y]) - \{\mu(x), \mu(y)\} \tag{2.10}$$

for $x, y \in \mathfrak{g}$.

Lemma 2.4.18. *$d_\mathfrak{g} c = 0$, i.e., c is a 2-cocycle in the Lie algebra cohomology of $\mathfrak{g}$.*

Proof. We have

$$\begin{aligned}
d_\mathfrak{g} c(x,y,z) =& - c([x,y],z) + c([x,z],y) - c([y,z],x) \\
=& - \mu([[x,y],z]) + \{\mu([x,y]), \mu(z)\} + \mu([[x,z],y]) - \{\mu([x,z]), \mu(y)\} \\
& - \mu([[y,z],x]) + \{\mu([y,z]), \mu(x)\} \\
=& \{\{\mu(x), \mu(y)\} + c(x,y), \mu(z)\} - \{\{\mu(x), \mu(z)\} + c(x,z), \mu(y)\} \\
& + \{\{\mu(y), \mu(z)\} + c(y,z), \mu(x)\} \\
=& \{\{\mu(x), \mu(y)\}, \mu(z)\} - \{\{\mu(x), \mu(z)\}, \mu(y)\} + \{\{\mu(y), \mu(z)\}, \mu(x)\} \\
=& 0,
\end{aligned}$$

since the Poisson bracket of any function with a constant vanishes. □

[10] I.e., we have a weak moment map.

Thus, c defines a cohomology class in $H^2(\mathfrak{g})$. In the previous section we saw that a different weak moment map μ' for the same action differs from μ by an element b of $\mathfrak{g}^*$, i.e., $\mu' = \mu + b$ for $b \in \mathfrak{g}^*$. Thus, choosing μ' results in

$$\begin{aligned} c'(x,y) :&= \mu'([x,y]) - \{\mu'(x), \mu'(y)\} \\ &= \mu([x,y]) + b([x,y]) - \{\mu(x) + b(x), \mu(y) + b(y)\} \\ &= \mu([x,y]) + b([x,y]) + \{\mu(x), \mu(y)\}, \end{aligned}$$

since $b(x)$ is a constant function for any $x \in \mathfrak{g}$, and its Poisson bracket with any function vanishes. It follows that choosing a different weak moment map $\mu' = \mu + b$ results in $c' = c - d_{\mathfrak{g}} b$, i.e., in the same cohomology class $[c]$. Therefore, we get the following:

Proposition 2.4.19. *Let $\rho : \mathfrak{g} \to \mathfrak{X}_{Ham}(M)$ be an action of a Lie algebra $\mathfrak{g}$ on a symplectic manifold (M, ω), and let $h : C^\infty(M) \to \mathfrak{X}_{Ham}$ be the map assigning a Hamiltonian vector field to a function. Then there exists a Lie algebra homomorphism $\mu : \mathfrak{g} \to C^\infty(M)$ such that $h \circ \mu = \rho$ iff $[c] = 0$ for c defined in 2.10*

Proof. Since the choice of another moment map changes c to $c' = c - d_{\mathfrak{g}} b$, then, if there exists a μ that is a Lie algebra homomorphism, i.e., $c = 0$ for that moment map, then $c' = -d_{\mathfrak{g}} b$ for any other choice of a moment map, i.e., $[c] = [c'] = 0$.

Conversely, suppose $[c] = 0$, i.e., $c = d_{\mathfrak{g}} a$ for some μ and $a \in \mathfrak{g}^*$ Then $\mu' = \mu + a$ is the desired Lie algebra homomorphism. □

The following corollary guarantees existence of moment maps based on a condition on Lie algebra cohomology of $\mathfrak{g}$.

Corollary 2.4.20. *Let $\mathfrak{g}$ be a Lie algebra of a connected Lie group G that acts on (M, ω) by Hamiltonian vector fields. If $H^2(\mathfrak{g}) = 0$, then this action admits a moment map.*

Example 2.4.21. Semi-simple Lie algebras.

For any semi-simple Lie algebra $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$ [11], therefore, by Corollaries 2.4.16 and 2.4.20, any action of $\mathfrak{g}$ admits a unique moment map.

The next 2 examples have $H^2(\mathfrak{g}) \neq 0$.

[11] This is a special case of the Whitehead lemmas, see, e.g., [32].

Example 2.4.22. Translations of $\mathbb{R}^{2n}$.

Let $\mathbb{R}^2$ act on $(\mathbb{R}^2, \omega = dq \wedge dp)$ by translations, i.e.,

$$(a_1, a_2) \cdot (q, p) := (a_1 + q, a_2 + p).$$

The infinitesimal generator corresponding to (a_1, a_2) is $-a_1 \frac{\partial}{\partial q} - a_2 \frac{\partial}{\partial p}$. Then

$$-\iota_{-a_1 \frac{\partial}{\partial q} - a_2 \frac{\partial}{\partial p}} dq \wedge dp = a_1 dp - a_2 dq = d(a_1 p - a_2 q),$$

i.e., the generators of the action are Hamiltonian. However, this action does not admit a moment map. Indeed, since $\mathbb{R}^2$ is abelian, $[c] = 0$ means $c = 0$. But

$$\begin{aligned} c(a, a') =& \mu[a, a'] - \{\mu(a), \mu(a')\} \\ =& -\{\mu(a), \mu(a')\} \\ =& -\omega(-a_1 \frac{\partial}{\partial q} - a_2 \frac{\partial}{\partial p}, -a_1' \frac{\partial}{\partial q} - a_2' \frac{\partial}{\partial p}) \\ =& a_1 a_2' - a_2 a_1', \end{aligned}$$

which does not vanish for linearly independent a and a'.

This can be straightforwardly generalized to the following statement:

$\mathbb{R}^{2n}$ acting on $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dq_i \wedge dp_i)$ by translations does not admit a moment map.

Remark 2.4.23. Note the difference between the above example and the Example 2.4.9, where $\mathbb{R}^n$ was acting on $\mathbb{R}^{2n}$.

Example 2.4.24. The Galilean group. ([2] [23] [46] [57]) The group of Galilean transformations

$$\begin{bmatrix} & & & v_1 & a_1 \\ & R & & v_2 & a_2 \\ & & & v_2 & a_3 \\ 0 & 0 & 0 & 1 & \tau \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where $R \in SO(3)$, $(v_1, v_2, v_3) = v \in \mathbb{R}^3$, $(a_1, a_2, a_3) = a \in \mathbb{R}^3$, $\tau \in \mathbb{R}$

is the symmetry group of Newtonian mechanics. It acts on the Newtonian spacetime in the following way:

$$\begin{pmatrix} x \\ t \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} (Rx + vt + a) \\ t + \tau \\ 1 \end{pmatrix},$$

where $x \in \mathbb{R}^3$, $t \in \mathbb{R}$.

The Galilean group acts on the phase space $\mathbb{R}^6$ of a particle with mass m moving in $\mathbb{R}^3$ by

$$(q, p) \mapsto (Rq + \frac{\tau}{m} Rp + \tau v + a, Rp + mv).$$

It can be shown that this action preserves the canonical symplectic form $\omega = \sum_{i=1}^{3} dq_i \wedge dp_i$ on $\mathbb{R}^6$.

The Lie algebra of the Galilean group is given by elements

$$x = \begin{bmatrix} & & & u_1 & b_1 \\ & A & & u_2 & b_2 \\ & & & u_2 & b_3 \\ 0 & 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $A \in \mathfrak{so}(3)$, $(u_1, u_2, u_3) = u \in \mathbb{R}^3$, $(b_1, b_2, b_3) = b \in \mathbb{R}^3$, $s \in \mathbb{R}$.

Let (A, u, b, s) and (A', u', b', s') denote two such elements x and x' respectively. Then the Lie bracket of x and x' is given by

$$[x, x'] = ([A, A'], Au' - A'u, Ab' - A'b + s'u - su', 0)$$

(see [23, §17]).

Consider the elements $x = (0, u, b, 0)$ and $x' = (0, u', b', 0)$, i.e.,

$$x = \begin{bmatrix} 0 & 0 & 0 & u_1 & b_1 \\ 0 & 0 & 0 & u_2 & b_2 \\ 0 & 0 & 0 & u_2 & b_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, x' = \begin{bmatrix} 0 & 0 & 0 & u'_1 & b'_1 \\ 0 & 0 & 0 & u'_2 & b'_2 \\ 0 & 0 & 0 & u'_2 & b'_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Lie bracket of these elements vanishes, i.e., $[x, x'] = 0$. The fundamental vector fields corresponding to x and x' are $v_x = \sum_{i=1}^{3} -b_i \frac{\partial}{\partial q_i} - u_i m \frac{\partial}{\partial p_i}$ and $v_{x'} = \sum_{i=1}^{3} -b'_i \frac{\partial}{\partial q_i} - u'_i m \frac{\partial}{\partial p_i}$.

Since $H^1(\mathbb{R}^6) = 0$, we know that the action of G is at least weakly Hamiltonian. Let μ be a weak moment map for this action. If $[c] = 0$, then $c(x, x') = 0$, since $[x, x'] = 0$. On the other hand, $\mu(x) = \sum_{i=1}^{3} m u_i q_i - b_i p_i$, therefore

$$\{\mu(x), \mu(x')\} = m(u \cdot b' - u' \cdot b).$$

It can be shown (see, e.g., [23, §53]) that for the Lie algebra $\mathfrak{g}$ of the Galilean group, $dim H^2(\mathfrak{g}) = 1$, and the cohomology class $[c]$ is intimately related to the notion of mass. Namely, consider a physical system given by a symplectic manifold with a transitive action of the Galilean group that preserves the symplectic structure. If this action admits a weak moment map, we can choose it so that the co-cycle c is determined uniquely as $c = mc_0$ for c_0 defined by

$$c_0(x, x') = u \cdot b' - u' \cdot b,$$

for x and x' given by (A, u, b, s) and (A', u', b', s') respectively.

The parameter m is a constant which can be interpreted as the mass of the physical system. For more see [57] or [23].

2.4.3.3 Moment maps for central extensions

Let G act on a symplectic manifold (M, ω). If μ is a weak moment map for this action, but the class $[c]$ defined in (2.10) does not vanish, then there is no (equivariant) moment map for this action, and the comoment map $\mu^* : \mathfrak{g} \to C^\infty(M)$ cannot be chosen to be a Lie algebra morphism. However, there is always a Lie algebra morphism $\bar{\mu}^* : \mathfrak{g} \oplus \mathbb{R} \to C^\infty(M)$ from the central extension of $\mathfrak{g}$ corresponding to the co-cycle (2.10) (see [31]):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^\infty(M) & \xrightarrow{h} & \mathfrak{X}_{Ham}(M) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle \bar{\mu}^*} & & \uparrow{\scriptstyle \rho} & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathfrak{g} \oplus \mathbb{R} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0
\end{array}$$

The map $\bar{\mu}^*$ is a comoment map for the action of $\mathfrak{g} \oplus \mathbb{R}$ on (M, ω), where $\mathbb{R}$ acts trivially.

The bracket on $\mathfrak{g} \oplus \mathbb{R}$ is given by

$$[(x, r_1), (y, r_2)]_{\mathfrak{g}\oplus\mathbb{R}} := ([x, y]_{\mathfrak{g}}, c(x, y)).$$

The map $\bar{\mu}^*$ is given by

$$\bar{\mu}^*(x, r) := \mu^*(x) - r.$$

The homomorphism property is easily verified:

$$\begin{aligned}\{\bar{\mu}^*(x, r_1), \bar{\mu}^*(y, r_2)\} &= \{\mu(x) + r_1, \mu(y) + r_2\} \\ &= \{\mu(x), \mu(y)\} \\ &= \mu([x, y]_{\mathfrak{g}}) - c(x, y) \\ &= \bar{\mu}^*([x, y]_{\mathfrak{g}}, c(x, y)) \\ &= \bar{\mu}^*([(x, r_1), (y, r_2)]_{\mathfrak{g}\oplus\mathbb{R}}).\end{aligned}$$

Example 2.4.25. The central extension of the Galilean algebra from Example 2.4.24 by the cocycle $c(x, x') = mc_0$ is often referred to as the *Bargmann algebra* ([5]).

Chapter 3

Multisymplectic geometry

This chapter will focus on multisymplectic geometry. The material of this chapter is based on a number of references that are cited whenever used. The only original content in this chapter is the material in §3.3.1.1 and §3.4.2.1 which is based on ongoing work.

§3.2 introduces the necessary background on L_∞-algebras.

§3.3 gives an introduction to multisymplectic geometry.

§3.4 introduces homotopy moment maps, provides examples and investigates the questions of existence and uniqueness.

3.1 Introduction

We saw in the previous chapter that symplectic geometry is the mathematical framework of the Hamiltonian formulation of classical mechanics. In this chapter we will be concerned with multisymplectic geometry, which is the mathematical framework of the Hamiltonian formulation of classical field theory.

It is also possible to give a Hamiltonian formulation of classical field theory using symplectic geometry, and indeed this approach is often taken by physicists. Such a formulation is obtained by considering fields to be mechanical systems with an infinite number of degrees of freedom (see, e.g., [49, §2.3.2]). Since a field has to be specified at every point of spacetime, the phase space will be infinite-dimensional, which creates certain problems: e.g., infinite-dimensional manifolds are not locally compact, and on a tangent space level a linear operator on an infinite-dimensional vector space can be injective, but not surjective ([27]), etc. Another drawback of the symplectic formulation of classical field theory is that it breaks covariance, i.e., the equal treatment of space and time variables, which is crucial for relativistic theories.[1]

More precisely, let S be the spacetime, and let $\Sigma \subset S$ be a certain hypersurface (called a *Cauchy surface*). The (instantaneous) *configuration space* of a given field theory is the space of all smooth sections $\Gamma(P)$ of a specified vector bundle $\pi : P \to \Sigma$. A field is then a section of $\pi : P \to \Sigma$ "evolving in time", and a solution of the field equations is a trajectory in $\Gamma(P)$. The Hamiltonian formulation is then defined on the cotangent bundle $T^*\Gamma(P)$ which has a canonical symplectic structure ([20]).

Thus, the symplectic formulation of classical field theory is formulated on an infinite-dimensional phase space and violates the spacetime covariance. The multisymplectic formulation of classical field theory allows to overcome the aforementioned problems. Instead of using time as the parameter space for the theory, multisymplectic formalism uses the whole spacetime. More precisely, consider spacetime S of dimension n and a vector bundle $\pi : E \to S$. A field ϕ is a section of $\pi : E \to S$, thus the fiber of $\pi : E \to S$ over any point $s \in S$ is the space of all possible values of the field at that point. We call $\pi : E \to S$ the *configuration bundle* of the theory. The *multiphase space* M of the theory is then defined to be the certain subbundle $\wedge_1^n T^*E$ of $\wedge^n T^*E$ which is equipped with a canonical $(n+1)$-form ω (for more see Example 3.3.6 and Remark 3.3.7 in §3.3). The Hamilton's equations can be formulated as

$$-\iota_{v_f}\omega = df,$$

where $v_f \in \Gamma(\wedge^n TM)$ is an n-vector field on M, and $f \in C^\infty(M)$.

The need for an $n+1$-form in this formulation of field theory as opposed to a 2-form in symplectic geometry is based on the following: in symplectic geometry, since we parameterize the theory by time, the solutions of field equations are curves in the configuration space. Since in field theory we parameterize by n-dimensional spacetime, the solutions are "n-dimensional curves", i.e., sections of the configuration bundle. Since in symplectic geometry curves in the configuration space correspond to curves in the phase space that are tangent to the Hamiltonian vector field, the solutions of field equations should correspond to the n-dimensional images of the sections that are tangent to Hamiltonian n-vector fields. A 2-form associates vector fields to Hamiltonian functions, while an $(n+1)$-form is needed to associate an n-vector field to a Hamiltonian function.

[1] These two problems don't arise in classical mechanics, since the base manifold is just $\mathbb{R}$, i.e., time, and there are no "space" coordinates.

3.2 L_∞-algebras

In this chapter we will see that each multisymplectic manifold has a certain L_∞-algebra associated to it. This section introduces L_∞ algebras and the corresponding notions and properties that will be needed for the purposes of this thesis.

3.2.1 Introduction and definition

As with many influential ideas in science, it is not easy to pinpoint the precise point in history when the idea of L_∞- structures first appeared. According to J. Stasheff [58, §4], the idea of "Jacobi up to homotopy" was implicitly present in homotopy theory starting from the early 1950's. As for the emergence of L_∞-structures in physics, according to the same paper, "In 1982, L_∞-algebras appeared in disguise in gravitational physics in work of D'Auria and Fré ... In 1989, the L_∞-structure of closed string field theory was first identified when Zwiebach gave a talk in Chapel Hill at the Grand Unification Theory workshop" (see [58, §4]). As a clear mathematical concept, the L_∞-algebras were introduced in [39]. Since their introduction, L_∞-algebras have been ubiquitous in mathematics as well as in theoretical physics. One of the most important appearances of L_∞-algebras has been in the proof of the celebrated "Formality theorem" by M. Kontsevich, which implies that every Poisson manifold admits a deformation quantization (see [37]).

We first briefly recall some notions we will need to define L_∞-algebras and morphisms between them. We begin by defining graded vector spaces.

Definition 3.2.1. Let $\mathbb{Z}$ be the set of integers. A *$\mathbb{Z}$-graded vector space* or simply a *graded vector space* V is a direct sum of vector spaces V_i

$$V = \bigoplus_{i \in \mathbb{Z}} V_i.$$

An element $v \in V_i$ is said to be *homogeneous of degree* i. The degree of v is denoted by $|v|$.

We will only consider graded vector spaces such that all the V_i are finite-dimensional. The dual of a graded vector space is defined in the following way:

Definition 3.2.2. Let $V = \bigoplus_{i\in\mathbb{Z}} V_i$ be a graded vector space. The dual of V is the graded vector space

$$V^* = \bigoplus_{i\in\mathbb{Z}} (V^*)_i,$$

where $(V^*)_i = (V_{-i})^*$ for all $i \in \mathbb{Z}$.

We define the tensor product of graded vector spaces:

Definition 3.2.3. Let V and W be graded vector spaces. The tensor product $V \otimes W$ is defined as

$$V \otimes W = \bigoplus_{i\in\mathbb{Z}} (V \otimes W)_i,$$

where $(V \otimes W)_i := \bigoplus_{i=k+l} V_k \otimes W_l$

Next we define the Koszul sign of a permutations of elements of a graded vector space. The following definition is taken from [52].

Definition 3.2.4. Let S_n denote the group of permutations of n elements, and let $\sigma \in S_n$ act on elements $v_1, ..., v_n$ of a graded vector space V. Let $(v_{i_1}, ..., v_{i_k})$ be the ordered subset of $v_1, ..., v_n$ consisting of all elements of $v_1, ..., v_n$ that have an odd degree. There is a unique permutation $\sigma' \in S_k$ such that $(v_{i_{\sigma'(1)}}, ..., v_{i_{\sigma'(k)}})$ is an ordered subset of $v_{\sigma(1)}, ..., v_{\sigma(n)}$ consisting of all elements that have an odd degree. Then the *Koszul sign* of σ acting on $v_1, ..., v_n$ is defined by

$$\epsilon(\sigma) := (-1)^{\sigma'},$$

where $(-1)^{\sigma'}$ denotes the sign of the permutation σ'.

Remark 3.2.5. Note that the Koszul sign defined above depends not only on the permutation σ, but on the $v_1, ..., v_n$ that the permutation is acting on, so a more precise notation for it would be $\epsilon(\sigma, v_1, ..., v_n)$. However, we avoid this notation due to the formulae in this chapter being quite cumbersome, so we shorten the notation whenever we can.

We can now define what it means for maps between graded vector spaces to be symmetric or skew-symmetric.

Definition 3.2.6. Let V and W be graded vector spaces, and let $V^{\otimes n}$ denote the n-fold tensor product of V. The map $f : V^{\otimes n} \to W$ is called *(graded) symmetric* if

$$f(v_{\sigma(1)}, ..., v_{\sigma(n)}) = \epsilon(\sigma) f(v_1, ..., v_n)$$

for all permutations $\sigma \in S_n$.

The map $f : V^{\otimes n} \to W$ is called *(graded) skew-symmetric* if

$$f(v_{\sigma(1)}, ..., v_{\sigma(n)}) = (-1)^{\sigma} \epsilon(\sigma) f(v_1, ..., v_n)$$

for all permutations $\sigma \in S_n$.

The following definition will be important when we define L_∞-algebras.

Definition 3.2.7. Let $\sigma \in S_n$ be a permutation, and let $p, q \in \mathbb{N}$ be such that $p+q=n$. The permutation σ is called a (p,q)*-unshuffle* if $\sigma(i) < \sigma(i+1)$ for all $i \neq p$. We will denote the set of (p,q)-unshuffles by $S(p,q)$.

The following notion will be important when we define morphisms between L_∞-algebras. The following definition is from [52].

Definition 3.2.8 (k-th graded symmetric power of V)**.** Let V be a graded vector space. Define the *k-th graded symmetrization operator* $Sym_k : V^{\otimes k} \to V^{\otimes k}$ by

$$Sym_k(v_1 \otimes ... \otimes v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon(\sigma) v_{\sigma(1)} \otimes ... \otimes v_{\sigma(k)},$$

where S_k denotes the group of permutations of k elements, and $\epsilon(\sigma)$ is the Koszul sign of σ acting on $v_1, ..., v_k$. Then the image of Sym_k is called the k-th graded symmetric power of V. We will denote this vector space by $S^k(V)$. We will also denote $Sym_k(v_1 \otimes ... \otimes v_k)$ just by concatenation $v_1...v_k$.

Finally, the following sign will often appear throughout this and the next chapters, so we introduce it here. For $k \in \mathbb{N}$, we define

$$\zeta(k) := -(-1)^{\frac{k(k+1)}{2}} \tag{3.1}$$

We can now define L_∞-algebras.

Definition 3.2.9. [39] An L_∞*-algebra* (or a *strong homotopy Lie algebra*) is a graded vector space L equipped with a collection

$$\{l_k : L^{\otimes k} \to L | 1 \leq k < \infty\}$$

of graded skew-symmetric linear maps (also called *multibrackets*) with $|l_k| = 2-k$, such that the following identity holds for $1 \leq m < \infty$:

$$\sum_{i+j=m+1} (-1)^{i(j-1)} \sum_{\sigma \in S(i,m-i)} (-1)^{\sigma} \epsilon(\sigma) l_j(l_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)}, ..., x_{\sigma(m)}) = 0, \tag{3.2}$$

where σ runs through $(i, m-i)$-unshuffles, and $\epsilon(\sigma)$ is the Koszul sign of σ acting on $x_1, ..., x_m$.

We will refer to (3.2) as the *generalized Jacobi identity* for reasons that will become clear further.

Unraveling the equation (3.2) for small values of m gives the following:

- For $m = 1$, we obtain $l_1 \circ l_1 = 0$. Thus, l_1 is a degree $2 - 1 = 1$ map that squares to 0, i.e., a differential.
- For $m = 2$, we obtain that l_1 is a graded derivation of l_2.
- For $m = 3$ the identity (3.2) means that, up to signs,

$$l_2(l_2(x,y),z) \pm l_2((x,z),y) \pm l_2((y,z),x) = \pm\, l_1(l_3(x,y,z)) \pm l_3(l_1(x),y,z) \pm l_3(l_1(y),x,z) \pm l_3(l_1(z),x,y),$$

i.e., the bilinear (graded) skew-symmetric map l_2 satisfies the (graded) Jacobi identity only up to the term on the right-hand side. If l_1 vanishes on x, y, z, we get that the Jacobi identity is satisfied up to $l_1(l_3(x,y,z))$. Since l_1 is a differential, it means that the Jacobi identity is satisfied up to an exact term or, using the language of homological algebra, "up to homotopy" (see [39]).

Thus, we get the following familiar examples of L_∞-algebras:

Example 3.2.10. A (co)chain complex $(L, d = l_1)$

$$\cdots \xrightarrow{d} L_{i-1} \xrightarrow{d} L_i \xrightarrow{d} L_{i+1} \cdots$$

Example 3.2.11. A differential graded Lie algebra $(L, l_1 = d, l_2 = \{\ ,\ \}, l_3 = 0)$

$$\cdots \xrightarrow{d} L_{i-1} \xrightarrow{d} L_i \xrightarrow{d} L_{i+1} \cdots$$

such that for $x, y \in L$

$$d\{x,y\} = d(x), y - (-1)^{|x||y|}\{dy, x\}$$

and

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0.$$

Note that, when L is concentrated in degree 0, and $l_1 = 0$, this becomes a Lie algebra.

In this thesis we will mostly encounter L_∞-algebras whose underlying graded vector spaces are finite direct sums of finite-dimensional vector spaces:

Definition 3.2.12. An L_∞-algebra concentrated in degrees $0, -1, ..., 1-n$ is called a *Lie n-algebra.* Since each l_k has degree $2-k$, this means that for a Lie n-algebra, $l_k = 0$ for $k > n+1$.

3.2.1.1 Central n-extensions

We would like to generalize the concept of central extensions of Lie algebras introduced in §2.2.4. In order to do this, we first state the following theorem from [3], a special case of which we will prove in Chapter 5.

Theorem 3.2.13. *[3, Thm.55] There is a one-to-one correspondence between Lie n-algebras with vanishing differential $l_1 = 0$ whose only nonzero terms are L_0 in degree zero and L_{1-n} in degree $1-n$, and quadruples $(\mathfrak{g}, V, \rho, c)$, where $\mathfrak{g}$ is a Lie algebra, V is a vector space, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ on V, and $c : \wedge^{n+1}\mathfrak{g} \to V$ is an $(n+1)$-cocycle in the Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in V.*

We now consider the Lie n-algebra corresponding to the Lie algebra $\mathfrak{g}$, the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$, and a cocycle $c : \wedge^{n+1}\mathfrak{g} \to \mathbb{R}$:

Definition 3.2.14. [7, §9.3] Let $\mathfrak{g}$ be a Lie algebra with the bracket $[\ ,\ ]_{\mathfrak{g}}$, and let $c : \wedge^{n+1}\mathfrak{g} \to \mathbb{R}$ be a cocycle in the Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in the trivial module $\mathbb{R}$.

A *central n-extension* of $\mathfrak{g}$ is a Lie n-algebra whose underlying vector space is $\mathfrak{g}$ in degree 0 and $\mathbb{R}$ in degree $1-n$, and whose brackets are given by

$$l_2(x_1, x_2) := [x_1, x_2]_{\mathfrak{g}}$$

$$l_{n+1}(x_1, ..., x_{n+1}) := c(x_1, ..., x_{n+1})$$

$$l_k \equiv 0 \ \text{ if } k \neq 2, k \neq n+1$$

when all $x_i \in \mathfrak{g}$, and are zero otherwise.

We will denote the central n-extension of $\mathfrak{g}$ corresponding to cocycle c by $\bar{\mathfrak{g}}_c$. Note that when $n = 1$, we recover the definition of a central extension of a Lie algebra $\mathfrak{g}$ by $\mathbb{R}$ (see §2.2.4) by setting $l_2 := [\ ,\ ]_{\mathfrak{g}} + c$.

3.2.2 Characterization as differential graded co-algebras

An alternative, more conceptual characterization of L_∞-algebras is given as follows. The exposition below closely follows [39] and [52, §2.2].

Let L be a graded vector space, and let sL denote L with the grading shifted by 1, i.e.,

$$(sL)_i := L_{i+1}. \tag{3.3}$$

Define maps $\tilde{l}_k : (sL)^{\otimes k} \to sL$ by

$$\tilde{l}_k(sx_1, ..., sx_k) = (-1)^{\alpha(x_1,...,x_k)} sl_k(x_1, ..., x_k)$$

where

$$\alpha(x_1, ..., x_k) = \begin{cases} \sum\limits_{i \text{ odd}} |x_i| & k \text{ even} \\ 1 + \sum\limits_{i \text{ even}} |x_i| & k \text{ odd.} \end{cases}$$

Note that these maps send an element of degree $\sum\limits_{i=1}^{k}(|x_k| - 1) = -k + \sum\limits_{i=1}^{k} |x_k|$, where $|x_k|$ denotes the degree of $x_k \in L$, to an element of degree $-1 + 2 - k + \sum\limits_{i=1}^{k} |x_k| = 1 - k + \sum\limits_{i=1}^{k} |x_k|$. Therefore, $\tilde{l}_k$ are maps of degree one. Moreover, they are graded-symmetric (see, e.g., [52, Lemma 2.6]). We can thus characterize the $\tilde{l}_k$ as linear maps $\tilde{l}_k : S^k(sL) \to sL$, where $S^k(sL)$ is the k-th (graded) symmetric power of sL. Consider $S^{\bullet \geq 1}(sL) = \bigoplus\limits_{k \geq 1} S^k(sL)$.

The vector space $S^{\bullet \geq 1}(sL) = \bigoplus\limits_{k \geq 1} S^k(sL)$ possesses a co-algebra structure, where the co-product is defined as follows (see, e.g., [52]):

$$\triangle(sx_1...sx_n) := \sum_{i=1}^{n-1} \sum_{\sigma \in S(i,n-i)} \epsilon(\sigma)(sx_{\sigma(1)}...sx_{\sigma(i)}) \otimes (sx_{\sigma(i+1)}...sx_{\sigma(n)}),$$

where $\epsilon(\sigma)$ is the Kozsul sign of σ acting on $sx_1, ..., sx_n$.

We can extend the maps $\tilde{l}_k$, using the same notation, as co-derivations to $\tilde{l}_k : S^{\bullet \geq 1}(sL) \to S^{\bullet \geq 1}(sL)$, using the co-algebra structure of $S^{\bullet \geq 1}(sL)$. Namely, we can define:

$$\tilde{l}_k(sx_1, sx_2, ..., sx_n) = \sum_{\sigma \in S(k,n-k)} \epsilon(\sigma)\tilde{l}_k(sx_{\sigma(1)}, ..., sx_{\sigma(k)})sx_{\sigma(k+1)}...sx_{\sigma(n)}, \tag{3.4}$$

where $\epsilon(\sigma)$ denotes the Kozsul sign of σ acting on $sx_1, ..., sx_n$, and $S(k, n-k)$ denotes the $(k, n-k)$-unshuffles.

Furthermore, we combine all $\tilde{l}_k$ into a degree 1 co-derivation

$$\tilde{l} := \sum \tilde{l}_l \tag{3.5}$$

We then have the following:

Proposition 3.2.15. *Let* $(L, \{l_k\})$ *be an* L_∞*-algebra, and* $(S^{\bullet \geq 1}(sL), \tilde{l})$ *be the corresponding co-commutative co-algebra together with the degree 1 co-derivation* $\tilde{l}$ *defined in* (3.5)*. Then the equations* (3.2) *are equivalent to* $\tilde{l}^2 = 0$*.*

Thus, L_∞-algebras can be equivalently characterized as co-commutative co-algebras with differentials. This allows for a more conceptual interpretation of L_∞-algebras, their generalized Jacobi identities, and morphisms of L_∞-algebras.

3.2.2.1 Chevalley-Eilenberg complex of an L_∞-algebra

Let $(L, \{l_k\})$ be an L_∞-algebra, and $S^{\bullet \geq 1}(sL)$ be the corresponding co-commutative co-algebra, as above.

By dualizing[2] the maps $\tilde{l}_k$ defined in (3.4), we obtain maps $d_i : S^{\bullet \geq 1}(sL)^* \to S^{\bullet \geq 1}(sL)^*$: for all $\xi \in S^k(sL)^*$,

$$d_i(\xi)(x_1...x_{k+i-1}) = \xi(\tilde{l}_i(x_1...x_{k+i-1})).$$

We can combine the d_i's into

$$d_{CE(L)} := d_1 - d_2 + d_3 - ... + d_{2k-1} - d_{2k} + \cdots.$$

Then $d_{CE(L)}$ is a differential on $S^{\bullet \geq 1}(sL)^*$, and $d^2_{CE(L)} = 0$ due to the generalized Jacobi identities (3.2) or, equivalently, $\tilde{l}^2 = 0$.

Note that we have taken the d_i's corresponding to the even indices of $\tilde{l}_i$ with a minus sign to match the conventional signs of the Chevalley-Eilenberg differential

[2] Assuming the L_i are finite-dimensional.

for Lie algebras. This does not affect $d^2_{CE(L)} = 0$ due to the quadratic nature of the Jacobi identities (3.2). We will illustrate this on the example of Lie 2-algebras below.

Definition 3.2.16. (see, e.g.,[56, §6])

The complex

$$CE(L) := (S^{\bullet \geq 1}(sL)^*, d_{CE(L)})$$

is the *Chevalley-Eilenberg complex* of the L_∞-algebra $(L, \{l_k\})$.

We will demystify this construction using the example of Lie 2-algebras, which we will need in Chapter 5:

The following example appeared in [44], with very similar wording.

Example 3.2.17. (**Chevalley-Eilenberg complex of a Lie 2-algebra**) Recall that a Lie 2-algebra is an L_∞-algebra concentrated in degrees $0, -1$. This means that the underlying graded vector space is of the form $\mathfrak{h} \oplus \mathfrak{g}$, where $\mathfrak{h}$ and $\mathfrak{g}$ are vector spaces, which we assume to be finite dimensional. The respective degrees of $\mathfrak{h}$ and $\mathfrak{g}$ in $\mathfrak{h} \oplus \mathfrak{g}$ are -1 and 0.

We denote the multibrackets l_1, l_2 and l_3, respectively, by $\delta, [\ ,\ ], [\ ,\ ,\ ]$. Namely, $\delta, [\ ,\ ], [\ ,\ ,\ ]$ are as follows:

$$\delta : \mathfrak{h} \to \mathfrak{g}$$

$$[\ ,\ ] : \wedge^2 \mathfrak{g} \to \mathfrak{g}$$

$$[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{h} \to \mathfrak{h}$$

$$[\ ,\ ,\ ] : \wedge^3 \mathfrak{g} \to \mathfrak{h}$$

and satisfy higher Jacobi identities (3.2) that are made explicit in [3, Lemma 33], namely:

$$\delta^2 = 0$$

$$\delta([x, h]) = [x, \delta h], \quad [\delta h, k] = [h, dk]$$

$$\delta([x, y, z]) = -[[x, y], z] + [[x, z], y] - [[y, z], x]$$

$$[\delta h, x, y] = -[[x, y], h] + [[x, h], y] - [[y, h], x]$$

and

$$[[u,x,y],z]+[[u,y,z],x]+[[u,y],x,z]+[[x,z],u,y]=[[u,x,z],y]+[[x,y,z],u]$$
$$+[[u,x],y,z]+[[u,z],x,y]$$
$$+[[x,y],u,z]+[[y,z],u,x]$$

for all $x,y,z,u \in \mathfrak{g}$, $h,k \in \mathfrak{h}$.

Consider the maps

$$\tilde{l}_1 : s\mathfrak{h} \to s\mathfrak{g} \qquad \tilde{l}_1(h) = -\delta(h)$$

$$\tilde{l}_2 : S^2(s\mathfrak{g}) \to s\mathfrak{g} \qquad \tilde{l}_2(x,y) = [x,y]$$

$$\tilde{l}_2 : s\mathfrak{g} \otimes s\mathfrak{h} \to s\mathfrak{h} \qquad \tilde{l}_2(x,h) = [x,h],\ \tilde{l}_2(h,x) = -[h,x]$$

$$\tilde{l}_3 : S^3(s\mathfrak{g}) \to s\mathfrak{h} \qquad \tilde{l}_3(x,y,z) = -[x,y,z]$$

Using (3.4), we extend the $\tilde{l}_i$ to maps $\tilde{l}_i : S^{\bullet\geq 1}(s(\mathfrak{h}\oplus\mathfrak{g})) \to S^{\bullet\geq 1}(s(\mathfrak{h}\oplus\mathfrak{g}))$ of degree 1, by the following formulae:

$$\tilde{l}_2(x_1x_2...x_n) = \sum_{\sigma\in S(2,n-2)} \epsilon(\sigma)\tilde{l}_2(x_{\sigma(1)},x_{\sigma(2)})x_{\sigma(3)}...x_{\sigma(n)}$$

and similarly for $\tilde{l}_1$ and $\tilde{l}_3$. Note that the x_i are elements of $s(\mathfrak{h}\oplus\mathfrak{g})$, and their respective degrees are the ones in $s(\mathfrak{h}\oplus\mathfrak{g})$; namely elements $x_i \in s\mathfrak{h}$ have degree -2, and elements $x_i \in s\mathfrak{g}$ have degree -1. By dualization, we obtain the following maps $d_i : S^{\bullet\geq 1}(s(\mathfrak{h}\oplus\mathfrak{g}))^* \to S^{\bullet\geq 1}(s(\mathfrak{h}\oplus\mathfrak{g}))^*$: for all $\xi \in S^k(s(\mathfrak{h}\oplus\mathfrak{g}))^*$,

$$d_1(\xi)(x_1...x_k) = \xi(\tilde{l}_1(x_1...x_k))$$

$$d_2(\xi)(x_1...x_{k+1}) = \xi(\tilde{l}_2(x_1...x_{k+1}))$$

$$d_3(\xi)(x_1...x_{k+2}) = \xi(\tilde{l}_3(x_1...x_{k+2})).$$

We combine the d_i into one map $d_{CE(L)} := d_1 - d_2 + d_3 : S^{\bullet\geq 1}(s(\mathfrak{h}\oplus\mathfrak{g}))^* \to S^{\bullet\geq 1}(s(\mathfrak{h}\oplus\mathfrak{g}))^*$ of degree 1. The fact that $d^2_{CE(L)} = 0$ follows from the generalized Jacobi identities (3.2) and, equivalently, $\tilde{l}^2 = 0$. Indeed,

$$(d_1 - d_2 + d_3)^2 = d_1^2 - d_1 d_2 - d_2 d_1 + d_2^2 + d_1 d_3 + d_3 d_1 - d_2 d_3 - d_3 d_2 + d_3^2,$$

i.e., $(d_1 - d_2 + d_3)^2 = 0$ is equivalent to

$$\begin{aligned} d_1^2 &= 0 \\ -d_1 d_2 - d_2 d_1 &= 0 \\ d_2^2 + d_1 d_3 + d_3 d_1 &= 0 \\ -d_2 d_3 - d_3 d_2 &= 0, \end{aligned}$$

which is equivalent to the even-numbered Jacobi identities (m being even in (3.2)) being multiplied by minus one.

The complex

$$CE(L) := (S^{\bullet \geq 1}(s(\mathfrak{h} \oplus \mathfrak{g}))^*, d_{CE(L)})$$

is the Chevalley-Eilenberg complex of the Lie 2-algebra $\mathfrak{h} \oplus \mathfrak{g}$.

3.2.3 L_∞-morphisms

Now that we have defined L_∞-algebras, we need to define morphisms between them. It would be tempting to define a morphism of L_∞-algebras as a linear map that preserves the multibrackets. Indeed, such a definition has its merits:

Definition 3.2.18. ([7, Def. 3.3]) Let (L, l_k) and (L', l'_k) be L_∞-algebras. A *strict L_∞-morphism* is a linear map of degree 0 $f : L \to L'$ such that

$$l'_k \circ f^{\otimes k} = f \circ l_k$$

for all $k \geq 1$.

However, this definition is, as the name suggests, too strict for many purposes. The results of the previous subsection let us define a morphism of L_∞-algebras in the following way:

Definition 3.2.19. ([7, Def. 3.4]) Let (L, l_k) and (L', l'_k) be L_∞-algebras, and let $(S^\bullet(sL), \tilde{l})$ and $(S^\bullet(sL'), \tilde{l}')$ be their corresponding differential graded co-algebras. An L_∞-morphism between (L, l_k) and (L', l'_k) is a morphism

$F : (S^\bullet(sL), \tilde{l}) \to (S^\bullet(sL'), \tilde{l}')$, i.e., a morphism $F : S^\bullet(sL) \to S^\bullet(sL')$ of graded co-algebras such that

$$F \circ \tilde{l} = \tilde{l}' \circ F.$$

Unraveling this definition reveals that an L_∞-morphism $f : L \to L'$ corresponds to a collection of (graded) skew-symmetric maps[3]

$$f_k : L^{\otimes k} \to L' \quad 1 \leq k < \infty$$

of degree $1-k$ that are "compatible with the brackets". We have taken the following explicit definition from [60, Def. 5] and [59, Def. 32]

Definition 3.2.20. Let (L, l_k) and (L', l'_k) be L_∞-algebras. A collection $\{f_k\}$ of graded skew-symmetric multilinear maps

$$f_k : L^{\otimes k} \to L'$$

of degree $1-k$ is an L_∞-morphism $\{f_k\} : (L, l_k) \to (L', l'_k)$ if and only if

$$K_f^k = 0 \quad \forall k \in \mathbb{N},$$

for $K_f^k : L^{\otimes k} \to L'$ defined by:

$$K_f^k(x_1, ..., x_k) := \sum_{i+j=k} (-1)^{ij} \sum_{\sigma \in S(i,j)} (-1)^\sigma \epsilon(\sigma) f_{j+1}(l_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)}, ..., x_{\sigma(i+j)})$$

$$-\sum_{m=1}^{k} \sum_{\substack{k_1+...+k_m=k \\ k_1 \leq ... \leq k_m}} (-1)^{\alpha(m,\bar{k})} \sum_{\sigma \in S_<(\bar{k})} (-1)^\tau \epsilon(\tau) l'_m(f_{k_1}(x_{\sigma(1)}, ..., x_{\sigma(k_1)}), ..., f_{k_m}(x_{\sigma(k-k_m+1)}, ..., x_{\sigma(k)}))$$

for all $x = (x_1, ..., x_k) \in L^{\otimes k}$, where

1. $\bar{k} := (k_1, ..., k_m)$ and $\alpha(m, \bar{k}) := m(m-1)/2 + k_1(m-1) + k_2(m-2) + ... + k_{m-1}$
2. $S(\bar{k})$ is the set of $\bar{k}$-unshuffles, i.e., permutations of $\{1, ..., k = k_1 + \cdots k_m\}$ such that

$$\sigma(k_1 + ... + k_{i-1} + 1) < ... < \sigma(k_1 + ... + k_{i-1} + k_i), \quad i = 1, ..., m$$

[3] We will slightly abuse notation and identify L_∞-morphisms with the corresponding maps $\{f_k\}$.

3. $S_<(\bar{k}) \subset S(\bar{k})$ is the set of $\bar{k}$-unshuffles such that
$$\sigma(k_1 + ... + k_{i-1} + 1) < \sigma(k_1 + ... + k_{i-1} + k_i + 1)$$
whenever $k_i = k_{i+1}$

4. $\bar{f} := (f_{k_1}, ..., f_{k_m}, x_1, ..., x_k)$, and τ is a permutation of $\{1, ..., m+k\}$ that sends $\bar{f}$ to
$$(f_{k_1}, x_{\sigma(1)}, ..., x_{\sigma(k_1)}, f_{k_2}, x_{\sigma(k_1+1)}, ..., x_{\sigma(k_1+k_2)}, ..., f_{k_m}, x_{\sigma(k-k_m+1)}, ..., x_{\sigma(k)}).$$

In particular, this definition implies that if $\{f_k\} : (L, l_k) \to (L', l'_k)$ is an L_∞-morphism, then f_1 is a morphism between the underlying cochain complexes (L, l_1) and (L', l'_1) of L and L'. i.e.,
$$f_1 \circ l_1 = l'_1 \circ f_1.$$

Note that strict morphisms as in Definition 3.2.18 then correspond to L_∞-morphisms $\{f_k\}$ where $f_k = 0$ for $k \geq 2$.

The next definition introduces a notion of equivalence of L_∞-algebras:

Definition 3.2.21. ([7, Def. 3.6]) An L_∞-morphism $\{f_k\} : (L, l_k) \to (L', l'_k)$ is called *an L_∞-quasi-isomorphism* if and only if the morphism of cochain complexes
$$f_1 : (L, l_1) \to (L', l'_1)$$
induces an isomorphism on the cohomology
$$[f_1] : H^\bullet(L) \to H^\bullet(L').$$

Finally, we present an explicit expression for compositions of L_∞-morphisms.

Definition 3.2.22. ([60, Def. 5], [59, Def. 32]) Let $f : (L, l_k) \to (L', l'_k)$ and $g : (L', l'_k) \to (L'', l''_k)$ be L_∞-morphisms. Then the composition $g \circ f : (L, l) \to (L'', l''_k)$ is an L_∞-morphism defined as $g \circ f := \{(g \circ f)_k\}$, where

$$(g \circ f)_k(x_1, ..., x_k) :=$$
$$\sum_{m=1}^{k} \sum_{\substack{k_1+...+k_m=k \\ k_1 \leq ... \leq k_m}} (-1)^{\alpha(m,\bar{k})} \sum_{\sigma \in S_<(\bar{k})} (-1)^\tau \epsilon(\tau) g_m(f_{k_1}(x_{\sigma(1)}, ..., x_{\sigma(k_1)}), ..., f_{k_m}(x_{\sigma(k-k_m+1)}, ..., x_{\sigma(x_k)}))$$

for all $x = (x_1, ..., x_k) \in L^{\otimes k}$, where $\bar{k}, \alpha(m, \bar{k}), S_<(\bar{k}), \tau$ and $\bar{f}$ are defined as in 3.2.20.

3.3 Multisymplectic manifolds

In this section we give a brief introduction to multisymplectic geometry, provide examples of multisymplectic manifolds and introduce concepts of central importance to this thesis.

3.3.1 Definition and examples

Definition 3.3.1. [9] A pair (M, ω) is an *n-plectic* manifold $(n \geq 1)$, if $\omega \in \Omega^{n+1}(M)$ is a closed nondegenerate $(n+1)$-form on M, i.e.,

$$d\omega = 0$$

and the map $\iota\_\omega : TM \to \wedge^n T^*M, v \mapsto \iota_v \omega$ is injective.

Example 3.3.2. For $n = 1$, we obtain the familiar example of symplectic manifolds.

Example 3.3.3. Any orientable manifold M of dimension $n+1$ equipped with a volume form ω is an example of an n-plectic manifold.

Example 3.3.4. ([4, Ex. 2.3]) (**Exterior powers of the cotangent bundle**) The following example generalizes Example 2.3.6.

Consider the n-th exterior power of the cotangent bundle $\wedge^n T^*Q$ of a manifold Q $(n \leq k = dim Q)$, with the projection map

$$\pi : \wedge^n T^*Q \to Q.$$

There is a canonical n-form θ on $\wedge^n T^*Q$ defined by

$$\theta_\eta(v_1, ..., v_n) := \eta(\pi_* v_1, ..., \pi_* v_n)$$

for $\eta \in \wedge^n T^*Q$ and $v_i \in T_\eta \wedge^n T^*Q$. Then $\omega = -d\theta$ is the *canonical n-plectic form* on $\wedge^n T^*Q$. The form ω is clearly closed. To see the non-degeneracy, it is helpful to write ω in coordinates: Let $q_1, ..., q_k$ be the coordinates on Q. The corresponding basis of n-forms on Q is given by $dq_I = dq_{i_1} \wedge \cdots \wedge dq_{i_n}$, where I runs through all indices of length n. Then θ is given by

$$\theta = \sum_I p_I dq_I,$$

where p_I are the fiber coordinates corresponding to dq_I, and ω is given by

$$\omega = -\sum_I dp_I \wedge dq_I.$$

Remark 3.3.5. Recall that the Darboux theorem (Theorem 2.3.5) in symplectic geometry states that any symplectic manifold is locally symplectomorphic to $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ with the canonical symplectic structure. This result does not generalize to n-plectic manifolds (M, ω) for $n \geq 2$, unless $n+1 = dimM$, i.e., ω is a volume form. However, certain Darboux-type results hold for special cases of multisymplectic manifolds. We refer the reader to [54, §4] for details.

Example 3.3.6. ([4, Ex. 2.4]) (**De Donder phase space**) Let $\pi : P \to M$ be a fiber bundle, and let $dimM = n$. For $p \in P$, $v \in T_pP$ is called *vertical* if $d\pi(v) = 0$. Let $\wedge_1^n T^*P \to P$ denote the bundle over P whose fiber at $p \in P$ consists of all n-forms $\alpha \in \wedge^n T_p^*P$ such that

$$\iota_{v_1}\iota_{v_2}\alpha = 0,$$

for all vertical $v_1, v_2 \in T_pP$. Note that $\wedge_1^n T^*P$ is a subbundle of $\wedge^n T^*P$. Let $i : \wedge_1^n T^*P \to \wedge^n T^*P$ denote the inclusion map, and let ω be the canonical n-plectic form on $\wedge^n T^*P$ constructed in the previous example. Then $i^*\omega$ is an n-plectic form on $\wedge_1^n T^*P$.

Remark 3.3.7. It can be shown (see, e.g., [21]) that $\wedge_1^n T^*P$ is isomorphic to the affine dual of the first jet bundle $\mathcal{J}^1P^*$. To see the connection with classical mechanics, consider a point moving in space Q. Take P to be the trivial bundle $P = M \times Q \to M$, where $M = \mathbb{R}$ represents time. Then sections of this bundle represent paths of the particle, and the first cojet bundle $\mathcal{J}^1P^*$ is symplectomorphic to the *extended phase space* $T^*(\mathbb{R} \times Q)$ of the particle.

Example 3.3.8. ([4, Ex. 2.2]) Let G be a compact simple Lie group. There exists an Ad-invariant inner product $\langle \, , \, \rangle$ on $\mathfrak{g}$ (see, e.g., [36, §4, Prop. 4.24]). Consider the following 3-form on G

$$\omega := \langle \theta^L, [\theta^L, \theta^L] \rangle,$$

where θ^L is a 1-form on G called the Maurer-Cartan form, given by $\theta^L_g : T_gG \to T_eG, v \mapsto L_{g^{-1}*}v$. This form is bi-invariant and so, as any bi-invariant form on a Lie group, is closed. It is also non-degenerate since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ for a semi-simple Lie algebra. Thus, (G, ω) is a 2-plectic manifold.

3.3.1.1 Generalization of coadjoint orbits

The following material is original and has not appeared in any papers or preprints yet.

Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h}$ a vector space; suppose we have a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ of a Lie algebra $\mathfrak{g}$, and a 3-cocycle c in the Lie algebra cohomology of $\mathfrak{g}$ with values in $\mathfrak{h}$. There is also an induced representation ρ^* of $\mathfrak{g}$ on the dual $\mathfrak{h}^*$: for $\xi \in \mathfrak{h}^*$, the element $\rho^*(x) \cdot \xi$ for $x \in \mathfrak{g}$ is given by

$$(\rho^*(x) \cdot \xi)(h) := -\xi(\rho(x) \cdot h) \tag{3.6}$$

for all $h \in \mathfrak{h}$.

By Proposition 2.2.5, there is also a representation $\phi : G \to GL(\mathfrak{h})$ of G on $\mathfrak{h}$, where G is the connected, simply-connected Lie group integrating $\mathfrak{g}$. There is also an induced representation ϕ^* of G on $\mathfrak{h}^*$, where for any $g \in G$, $(\phi^*(g) \cdot \xi)(h)$ is given by

$$(\phi^*(g) \cdot \xi)(h) := \xi(\phi(g^{-1}) \cdot h).$$

We will often omit the ϕ or ϕ^* and denote the action of an element $g \in G$ on $h \in \mathfrak{h}$ and $\xi \in \mathfrak{h}^*$ by $g \cdot h$ and $g \cdot \xi$ respectively.

We define a 3-form ω on an orbit σ_ξ of $\xi \in \mathfrak{h}^*$ under the action of G by

$$\omega_\xi(v_x, v_y, v_z) := \xi(c(x, y, z)), \tag{3.7}$$

where the v_x, v_y, v_z are the fundamental vector fields corresponding to x, y, z respectively: for $x \in \mathfrak{g}$, the vector field v_x at point $\xi \in \mathfrak{h}^*$ is given by

$$v_x|_\xi = \frac{d}{dt}|_{t=0} exp(-tx) \cdot \xi = \rho^*(-x) \cdot \xi, \tag{3.8}$$

where ρ^* is the dual representation of $\mathfrak{g}$ on $\mathfrak{h}^*$. Note that the form (3.7) is not necessarily well-defined.

Let $\mathfrak{g}_\xi$ denote the Lie algebra of the stabilizer of ξ under the action of G. Then, by Proposition 2.2.24, $\mathfrak{g}_\xi = \{x \in \mathfrak{g} : v_x|_\xi = 0\}$, i.e., $\mathfrak{g}_\xi = \{x \in \mathfrak{g} : \rho^*(-x) \cdot \xi = 0\}$. or equivalently, using the definition of the dual representation,

$$\mathfrak{g}_\xi = \{x \in \mathfrak{g} : \xi \circ \rho(x) = 0\}.$$

The form (3.7) is well-defined if

$$\mathfrak{g}_\xi \subset ker(\xi \circ c) \tag{3.9}$$

and non-degenerate if

$$\mathfrak{g}_\xi = ker(\xi \circ c). \tag{3.10}$$

Note that, since $c \in \wedge^3 \mathfrak{g}^* \otimes h$ and $\xi \in \mathfrak{h}^*$, we have $\xi \circ c \in \wedge^3 \mathfrak{g}^*$.

Definition 3.3.9. An element c of $\wedge^k \mathfrak{g}^* \otimes \mathfrak{h}$ is *G-invariant* if

$$g \cdot c(x_1, ..., x_k) = c(Ad_g x_1, ..., Ad_g x_k)$$

for all $g \in G$ and all $x_i \in \mathfrak{g}$.

Proposition 3.3.10. *Let $c \in \wedge^3\mathfrak{g}^* \otimes \mathfrak{h}$ be G-invariant. Then, if the condition* (3.10) *holds at one point of the orbit σ_ξ, it holds at all points of σ_ξ.*

Proof. We have to show that, if $ker(\xi \circ c) = \mathfrak{g}_\xi$, then $ker(g\xi \circ c) = \mathfrak{g}_{g\xi}$ for all $g \in G$. Note that $\mathfrak{g}_{g\xi} = Ad_g\mathfrak{g}_\xi$. First, we show that $\mathfrak{g}_{g\xi} = Ad_g\mathfrak{g}_\xi \subset ker(g\xi \circ c)$. For $\forall x \in \mathfrak{g}_\xi$, using the invariance of c, we get:

$$
\begin{aligned}
(g\xi \circ c)(Ad_g x, y, z) &= \xi(g^{-1} \cdot (c(Ad_g x, y, z))) = \xi(c(Ad_{g^{-1}}Ad_g x, Ad_{g^{-1}}y, Ad_{g^{-1}}z)) \\
&= \xi(c(x, Ad_{g^{-1}}y, Ad_{g^{-1}}z)) \\
&= 0.
\end{aligned}
$$

Now suppose $x \in ker(g\xi \circ c)$, we want to show $x \in \mathfrak{g}_{g\xi} = Ad_g\mathfrak{g}_\xi$. For all $y, z \in \mathfrak{g}$, by invariance of c, we have:

$$0 = (g\xi \circ c)(x, y, z) = \xi(g^{-1} \cdot c(x, y, z)) = \xi(c(Ad_{g^{-1}}x, Ad_{g^{-1}}y, Ad_{g^{-1}}z))$$

It follows, that for every $x \in ker(g\xi \circ c), Ad_{g^{-1}}x \in ker(\xi \circ c) = \mathfrak{g}_\xi$. Then $x = Ad_g \cdot Ad_{g^{-1}}x \in \mathfrak{g}_{g\xi}$. □

The following proposition shows that ω defined in (3.7) is closed.

Proposition 3.3.11. *Let ω be the 3-form defined in* (3.7)*. Then $d\omega = 0$.*

Proof. By the definition of $d\omega$, we have:

$$
\begin{aligned}
(d\omega)_\xi(v_x, v_y, v_z, v_u) &= (v_x \cdot \omega(v_y, v_z, v_u))_\xi - (v_y \cdot \omega(v_x, v_z, v_u))_\xi + (v_z \cdot (v_x, v_y, v_u))_\xi \\
&- (v_u \cdot \omega(v_x, v_y, v_z))_\xi - \omega_\xi([v_x, v_y], v_z, v_u) + \omega_\xi([v_x, v_z], v_y, v_u) \\
&- \omega_\xi([v_x, v_u], v_y, v_z) - \omega_\xi([v_y, v_z], v_x, v_u) + \omega_\xi([v_y, v_u], v_x, v_z) \\
&- \omega_\xi([v_z, v_u], v_x, v_y)
\end{aligned}
$$

Evaluating the first summand on the righ-hand side of the above expression gives: $(v_x \cdot \omega(v_y, v_z, v_u))_\xi = v_x \cdot \langle c(y, z, u), \xi\rangle$, where $\langle\ ,\ \rangle$ is the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. Note that $\langle c(y, z, u), \xi\rangle$ is a linear function of the argument ξ. For convenience we denote this function by $f : \mathfrak{h}^* \to \mathbb{R}$, i.e., $f(\xi) = \langle c(y, z, u), \xi\rangle$ for $\xi \in \mathfrak{h}^*$. Then,

$$v_x \cdot \langle c(y, z, u), \xi\rangle = v_x \cdot f(\xi) = (df)(v_x|_\xi) = f(v_x|_\xi),$$

where we have used that the derivative of a linear function f is the function itself, and that $v_x|_\xi \in \mathfrak{h}^*$, since v_x is a linear vector field on a vector space $\mathfrak{h}^*$. Therefore,

$$v_x \cdot \langle c(y, z, u), \xi\rangle = \langle c(y, z, u), v_x|_\xi\rangle = (\rho^*(-x) \cdot \xi)(c(y, z, u)) = \xi(\rho(x) \cdot c(y, z, u)),$$

where we have used (3.8) and the definition of the dual representation (3.6). Similarly, we get analogous expressions for the 3 other terms on the right-hand side of the first formula in the proof. Using this, the definition of ω_ξ, and the fact that $v_{[x,y]} = [v_x, v_y]$, we obtain:

$$\begin{aligned}(d\omega)_\xi(v_x, v_y, v_z, v_u) &= \xi(\rho(x)\cdot c(y,z,u)) - \xi(\rho(y)\cdot c(x,z,u)) + \xi(\rho(z)\cdot c(x,y,u)) \\ &- \xi(\rho(u)\cdot c(x,y,z)) - \xi(c([x,y],z,u)) + \xi(c([x,z],y,u)) \\ &- \xi(c([x,u],y,z)) - \xi(c([y,z],x,u)) + \xi(c([y,u],x,z)) \\ &- \xi(c([z,u],x,y)) \\ &= 0,\end{aligned}$$

since c is a cocycle for the Lie algebra cohomology of $\mathfrak{g}$ with values in $\mathfrak{h}$.

□

The next proposition shows that, if c is an invariant cocycle, then ω is G-invariant. We will first need the following short lemma.

Lemma 3.3.12. *$c \in \wedge^3\mathfrak{g}^* \otimes \mathfrak{h}$ is G-invariant if and only if the following holds for all $x, y, z, u \in \mathfrak{g}$:*

$$\rho(u)(c(x,y,z)) - c([u,x],y,z) + c([u,y],x,z) - c([u,z],x,y) = 0 \tag{3.11}$$

Proof. Assume c is G-invariant. Then

$$\begin{aligned}\rho(u)(c(x,y,z)) &= \frac{d}{dt}|_{t=0} exp(tu)\cdot c(x,y,z) \\ &= \frac{d}{dt}|_{t=0} c(Ad_{exp(tu)}x, Ad_{exp(tu)}y, Ad_{exp(tu)}z)) \\ &= c(ad_u x, y, z) + c(x, ad_u y, z) + c(x, y, ad_u z)\end{aligned}$$

The converse follows from G being connected. □

Proposition 3.3.13. *Let $c \in \wedge^3\mathfrak{g}^* \otimes \mathfrak{h}$ be invariant, and ω the corresponding 3-form defined in* (3.7)*. Then ω is invariant under the G-action on σ_ξ.*

Proof. Let $u \in \mathfrak{g}$ an v_u be the corresponding fundamental vector field of the action. Since G is connected, it suffices to prove that $\pounds_{v_u}\omega = 0$ for all $u \in \mathfrak{g}$.

By the Leibniz rule for the Lie derivative we have

$$\begin{aligned}(\pounds_{v_u}\omega)(v_x,v_y,v_z)_\xi &= \pounds_{v_u}(\omega(v_x,v_y,v_z))_\xi - \omega(\pounds_{v_u}v_x,v_y,v_z)_\xi \\ &- \omega(v_x,\pounds_{v_u}v_y,v_z)_\xi - \omega(v_x,v_y,\pounds_{v_u}v_z)_\xi \\ &= (v_u\cdot\omega(v_x,v_y,v_z))_\xi - \omega([v_u,v_x],v_y,v_z)_\xi \\ &- \omega(v_x,[v_u,v_y],v_z)_\xi - \omega(v_x,v_y,[v_u,v_z])_\xi.\end{aligned}$$

We showed in the proof of Proposition 3.3.11 that $(v_u\cdot\omega(v_x,v_y,v_z))_\xi$ equals $\xi(\rho(u)\cdot c(x,y,z))$. Using this and the definition of ω in (3.7), we see that the right-hand side becomes

$$\xi(\rho(u)\cdot c(x,y,z) - c([u,x],y,z) - c(x,[u,y],z) - c(x,y,[u,z]))$$

Using the invariance of c and (3.11), we obtain $\pounds_{v_u}\omega = 0$. □

Combining propositions (3.3.11) and (3.3.13), we obtain:

Proposition 3.3.14. *Let c be a G-invariant 3-cocycle in the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with values in $\mathfrak{h}$. Consider the action of G on $\mathfrak{h}^*$, and let σ_ξ be the orbit of $\xi\in\mathfrak{h}^*$ under this action. Assume $\mathfrak{g}_\xi = ker(\xi\circ c)$, and let ω be a 3-form on σ_ξ defined by* (3.7)*. Then (σ_ξ,ω) is a 2-plectic manifold with a G-invariant 2-plectic form.*

We can consider the special case of the above, where $\mathfrak{h} = \wedge^2\mathfrak{g}$, the representation of $\mathfrak{g}$ on $\mathfrak{h} = \wedge^2\mathfrak{g}$ is the adjoint representation

$$ad_x(y\wedge z) = ad_x y\wedge z + x\wedge ad_x z,$$

and the cocycle c is given by

$$c = d_{\mathfrak{g}} Id_{\wedge^2\mathfrak{g}},$$

where $Id_{\wedge^2\mathfrak{g}}$ is the identity map $\wedge^2\mathfrak{g}\to\wedge^2\mathfrak{g}, x\wedge y\mapsto x\wedge y$, and $d_{\mathfrak{g}}$ is the differential in the Chevalley Eilenberg cohomology of $\mathfrak{g}$ with values in the representation $\wedge^2\mathfrak{g}$. The 2-cochain $Id_{\wedge^2\mathfrak{g}}\in\wedge^2\mathfrak{g}^*\otimes\wedge^2\mathfrak{g}$ is G-invariant, since

$$\begin{aligned}ad_u\cdot(Id_{\wedge^2\mathfrak{g}}(x,y)) &= ad_u\cdot(x\wedge y) = (ad_u x\wedge y) + (x\wedge ad_u y) \\ &= Id_{\wedge^2\mathfrak{g}}(ad_u x,y) + Id_{\wedge^2\mathfrak{g}}(x,ad_u y)\end{aligned}$$

by the definition of the adjoint action. We will see later in this chapter, namely in Lemma 3.4.7, that this implies G-invariance of $c = d_{\mathfrak{g}} Id_{\wedge^2\mathfrak{g}}$.

Thus, for $\xi \in \wedge^2\mathfrak{g}^*$ we consider the 2-plectic form ω form on the orbit of ξ given by

$$\begin{aligned}
\omega_\xi(v_x, v_y, v_z) &= (\xi \circ d_{\mathfrak{g}} Id_{\wedge^2\mathfrak{g}})(x, y, z) \\
&= \xi(ad_x(y \wedge z)) - \xi(ad_y(x \wedge z)) + \xi(ad_z(x \wedge y)) \\
&- \xi([x, y], z) + \xi([x, z], y) - \xi([y, z], x) \\
&= \xi([x, y], z) + \xi(y, [x, z]) - \xi([y, x], z) - \xi(x, [y, z]) \\
&+ \xi([z, x], y) + \xi(x, [z, y]) - \xi([x, y], z) + \xi([x, z], y) - \xi([y, z], x) \\
&= \xi([x, y], z) - \xi([x, z], y) + \xi([y, z], x).
\end{aligned}$$

Note that, up to sign, the last expression on the right-hand side coincides with the Chevalley-Eilenberg differential of $\xi \in \wedge^2\mathfrak{g}^*$ with values in the trivial representation, evaluated at $x, y, z \in \mathfrak{g}$.

Remark 3.3.15. Note that this generalizes the symplectic structure on the coadjoint orbits (Example 2.3.7) of $\mathfrak{g}$, where

$$\omega_\xi(v_x, v_y) = \xi([x, y]).$$

Indeed, consider the adjoint representation of $\mathfrak{g}$ on itself and the Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in this representation. Then for the 1-cochain $Id_g : \mathfrak{g}^* \otimes \mathfrak{g}$ given by $Id_{\mathfrak{g}}(x) = x$, we have:

$$\begin{aligned}
d_{\mathfrak{g}} Id_{\mathfrak{g}}(x, y) &= ad_x \cdot Id_{\mathfrak{g}}(y) - ad_y \cdot Id_{\mathfrak{g}}(x) - Id_{\mathfrak{g}}([x, y]) \\
&= [x, y] - [y, x] - [x, y] \\
&= [x, y],
\end{aligned}$$

so $(\xi \circ d_{\mathfrak{g}} Id_{\mathfrak{g}})(x, y) = \xi([x, y])$ and $\omega_\xi(v_x, v_y) = (\xi \circ d_{\mathfrak{g}} Id_{\mathfrak{g}})(x, y)$

However, unlike the 2-plectic case, the well-definedness and nondegeneracy of ω are automatic in the symplectic case. Namely, the condition

$$\mathfrak{g}_\xi = ker(\xi \circ c)$$

becomes

$$\mathfrak{g}_\xi = ker(\xi \circ d_{\mathfrak{g}} Id_{\mathfrak{g}})$$

which holds at all points $\xi \in \mathfrak{g}^*$, since $\mathfrak{g}_\xi = \{x \in \mathfrak{g} | v_x|_\xi = 0\}$ (Proposition 2.2.24) and $\langle v_x|_\xi, y\rangle = \langle \xi, [x, y]\rangle$ for the coadjoint action of G on $\mathfrak{g}^*$ (Example 2.2.17).

Meanwhile, for $\xi \in \wedge^2\mathfrak{g}^*$

$$\begin{aligned}\langle v_x|_\xi, y \wedge z\rangle &= \frac{d}{dt}\Big|_{t=0} \langle Ad_{exp(-tx)}\xi, y \wedge z\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle \xi, Ad_{exp(tx)}(y \wedge z)\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle \xi, Ad_{exp(tx)}y \wedge Ad_{exp(tx)}z)\rangle \\ &= \langle \xi, [x,y] \wedge z + y \wedge [x,z]\rangle,\end{aligned}$$

so $x \in \mathfrak{g}_\xi$ iff

$$\langle \xi, [x,y] \wedge z + y \wedge [x,z]\rangle = 0 \tag{3.12}$$

for all $y, z \in \mathfrak{g}$.

On the other hand, $x \in ker(\xi \circ c) = ker(\xi \circ d_{\mathfrak{g}} Id_{\wedge^2 \mathfrak{g}})$ iff

$$\langle \xi, [x,y] \wedge z - [x,z] \wedge y + [y,z] \wedge x\rangle. \tag{3.13}$$

Therefore, comparing formulae (3.12) and (3.13), we see that they are different, and therefore the condition $\mathfrak{g}_\xi = ker(\xi \circ c)$ is nontrivial.

For example, if $x \in \mathfrak{g}_\xi$ then $x \in ker(\xi \circ d_{\mathfrak{g}} Id_{\wedge^2 \mathfrak{g}})$ iff $\langle \xi, x \wedge [y,z]\rangle = 0$ for all $y, z \in \mathfrak{g}$.

Below we present examples of Lie algebras $\mathfrak{g}$ and $\xi \in \wedge^2 \mathfrak{g}^*$ where the condition $\mathfrak{g}_\xi = ker(\xi \circ d_{\mathfrak{g}} Id_{\wedge^2 \mathfrak{g}})$ holds.

Example 3.3.16. Let $\mathfrak{g}$ be the 3-dimensional Lie algebra with the bracket relations between the basis elements given by $[x,y] = 0, [x,z] = x, [y,z] = y$. The point $\xi = x^* \wedge y^* \in \wedge^2 \mathfrak{g}^*$ satisfies $\mathfrak{g}_\xi = 0$.

Furthermore, calculating $\xi \circ d_{\mathfrak{g}}\mathrm{Id}$ on basis elements gives $\xi \circ d_{\mathfrak{g}}\mathrm{Id} = -2x^* \wedge y^* \wedge z^*$, so clearly $ker(\xi \circ d_{\mathfrak{g}}\mathrm{Id}) = 0 = \mathfrak{g}_\xi$, and, therefore, the form ω is well-defined and nondegenerate.

Example 3.3.17. Let $\mathfrak{g}$ be the 3-dimensional Lie algebra with the bracket relations between the basis elements given by $[x,y] = 0, [x,z] = y, [y,z] = x + y$. The point $\xi = x^* \wedge y^* \in \wedge^2 \mathfrak{g}^*$ satisfies $\mathfrak{g}_\xi = 0$.

Furthermore, calculating $\xi \circ d_{\mathfrak{g}}\mathrm{Id}$ on basis elements gives $\xi \circ d_{\mathfrak{g}}\mathrm{Id} = -x^* \wedge y^* \wedge z^*$, so clearly $ker(\xi \circ d_{\mathfrak{g}}\mathrm{Id}) = 0 = \mathfrak{g}_\xi$, and the form ω is well-defined and nondegenerate.

The following example is taken from [41].

Example 3.3.18. ([41, Ex. 5.7]). Let $\mathfrak{g} = \mathfrak{su}(3)$ be the Lie algebra of complex matrices with the following basis: $A_j = i(E_{jj} - E_{j+1,j+1}), B_{kl} = E_{kl} - E_{lk}, C_{kl} = i(E_{kl} + E_{lk})$, for $j, k = 1, 2, k < l = 2, 3$, where E_{pq} is an elementary 3 by 3 matrix with 1 at position (p, q). Let $A_j^*, B_{kl}^*, C_{kl}^*$ denote the dual basis.

Take $\xi = B_{12}^* \wedge B_{13}^* - C_{12}^* \wedge C_{13}^*$. By direct computation, one finds that $\mathfrak{g}_\xi =< A_2, B_{23}, C_{23} >$ and that $(\xi \circ d_{\mathfrak{g}}\mathrm{Id}) = 3A_1^* \wedge (B_{13}^* \wedge C_{12}^* - B_{12}^* \wedge C_{13}^*)$, so $ker(\xi \circ d_{\mathfrak{g}}\mathrm{Id}) = \mathfrak{g}_\xi$. Therefore, the form ω is well-defined and nondegenerate.

Example 3.3.19. Let $\mathfrak{g}$ be the 4-dimensional Lie algebra with the following bracket relations among the basis elements $x, y, z, u \in \mathfrak{g}$: $[x, y] = x, [z, u] = 0$, all other brackets being 0. The points $\xi = x^* \wedge y^* + x^* \wedge z^* + x^* \wedge u^*$ and $\xi' = x^* \wedge z^* + y^* \wedge z^* + u^* \wedge z^*$ satisfy $\mathfrak{g}_\xi = \mathfrak{g}_{\xi'} = 0$, thus $\mathfrak{g}_\xi \subset (\xi \circ d_{\mathfrak{g}}Id)$, $\mathfrak{g}_{\xi'} \subset (\xi' \circ d_{\mathfrak{g}}Id)$, and ω is well-defined on both σ_ξ and $\sigma\xi'$. However,

$$\xi \circ d_{\mathfrak{g}}\mathrm{Id} = x^* \wedge y^* \wedge z^* + x^* \wedge y^* \wedge u^* - x^* \wedge z^* \wedge u^*$$

and

$$\xi' \circ d_{\mathfrak{g}}\mathrm{Id} = x^* \wedge y^* \wedge z^* - x^* \wedge z^* \wedge u^* - y^* \wedge z^* \wedge u^*,$$

and both of them are degenerate. Therefore, the forms ω_ξ and $\omega_{\xi'}$ defined respectively on σ_ξ and $\sigma_{\xi'}$ are well-defined, but degenerate.

3.3.2 Hamiltonian forms

Definition 3.3.20. A vector field $v \in \mathfrak{X}(M)$ on an n-plectic manifold (M, ω) is a *multisymplectic vector field* if

$$\mathcal{L}_v\omega = 0.$$

We will denote the set of multisymplectic vector fields by $\mathfrak{X}_{MSympl}(M)$

The following lemma is fully analogous to Lemma 2.3.10, and the proof is exactly the same.

Lemma 3.3.21. *Let (M, ω) be an n-plectic manifold. A vector field $v \in \mathfrak{X}(M)$ is a multisymplectic vector field iff $\iota_v\omega$ is closed.*

Definition 3.3.22. An $(n-1)$-form α on an n-plectic manifold (M, ω) is *Hamiltonian* iff there exists a vector field $v_\alpha \in \mathfrak{X}(M)$ such that

$$d\alpha = -\iota_{v_\alpha}\omega.$$

The vector field v_α is the *Hamiltonian vector field* corresponding to α.

We will denote the set of Hamiltonian vector fields by $\mathfrak{X}_{Ham}(M)$ and the set of Hamiltonian $(n-1)$-forms by $\Omega^{n-1}_{Ham}(M)$.

Remark 3.3.23. Note that, unlike in symplectic geometry, where due to ω inducing an isomorphism between TM and T^*M any $f \in C^\infty(M)$ was a Hamiltonian 0-form, not every $(n-1)$-form is a Hamiltonian form. However, due to the non-degeneracy of ω, the corresponding Hamiltonian vector field is unique for each $\alpha \in \Omega^{n-1}_{Ham}(M)$, as in symplectic geometry.

Just like in symplectic geometry, $v \in \mathfrak{X}(M)$ is a Hamiltonian vector field iff $i_v\omega$ is exact. Therefore, all Hamiltonian vector fields are also mutisymplectic, i.e., $\mathfrak{X}_{Ham}(M) \subset \mathfrak{X}_{MSympl}(M)$.

Again, as in symplectic geometry, we have the following:

Lemma 3.3.24. *The Lie bracket of multisymplectic vector fields is Hamiltonian, i.e.,*

$$[\mathfrak{X}_{MSympl}, \mathfrak{X}_{MSympl}] \subset \mathfrak{X}_{Ham}(M).$$

Corollary 3.3.25. $\mathfrak{X}_{Ham}(M)$ *is a Lie algebra ideal of* $\mathfrak{X}_{MSympl}(M)$.

The following useful formula is known as the "Extended Cartan Formula" and can be found in [42, Lemma 3.4].

Lemma 3.3.26. *Let* $\alpha \in \Omega^m(M)$. *Then for all* $k \geq 2$ *and all vector fields* $v_1, ..., v_k$, *we have:*

$$\begin{aligned}(-1)^k d\iota(v_1 \wedge ... \wedge v_k)\alpha = &\sum_{1 \leq i < j \leq k} (-1)^{i+j}\iota([v_i, v_j] \wedge v_1 \wedge ... \wedge \hat{v}_i \wedge ... \wedge \hat{v}_j \wedge ... \wedge v_k)\alpha \\ &+ \sum_{i=1}^{k} (-1)^i \iota(v_1 \wedge ... \wedge \hat{v}_i \wedge ... \wedge v_k)\pounds_{v_i}\alpha \\ &+ \iota(v_1 \wedge ... \wedge v_k)d\alpha,\end{aligned}$$

where $\iota(\dots)$ *denotes contraction with a multivector field:* $\iota(v_{\alpha_1} \wedge ... \wedge v_{\alpha_k})\omega = \iota_{v_{\alpha_k}} ... \iota_{v_{\alpha_1}}\omega$.

3.3.3 The Lie n-algebra of observables

Recall that the space of smooth functions $C^\infty(M)$ on a symplectic manifold (M, ω) had a Lie algebra structure, which we called the Lie algebra of observables, and the bracket was given by

$$\{f, g\} = \iota_{v_f \wedge v_g}\omega,$$

where v_f and v_g were the Hamiltonian vector fields of f and g.

We would like to similarly associate a Lie algebra of observables to an n-plectic manifold for $n > 1$. Since in the symplectic case the algebra of observables consisted of (Hamiltonian) functions, the natural choice in n-plectic geometry would be the space of Hamiltonian $(n-1)$-forms. Analogously to the symplectic case, we can define the bracket of Hamiltonian $(n-1)$-forms to be (see, e.g., [4, §3])

$$\{\alpha, \beta\} = \iota_{v_\alpha \wedge v_\beta}\omega.$$

Indeed, we have

Lemma 3.3.27. *For $\alpha, \beta \in \Omega^{n-1}_{Ham}(M)$,*

$$d\{\alpha, \beta\} = -\iota_{[v_\alpha, v_\beta]}\omega,$$

i.e., $\{\alpha, \beta\} \in \Omega^{n-1}_{Ham}(M)$, and the Hamiltonian vector field corresponding to $\{\alpha, \beta\}$ is $[v_\alpha, v_\beta]$.

Proof. Exactly as the proof of Lemma 2.3.15, noting that $d\iota_{v_\alpha \wedge v_\beta}\omega = d\iota_{v_\beta}\iota_{v_\alpha}\omega = -d\iota_{v_\alpha}\iota_{v_\beta}\omega$. □

This bracket is also manifestly skew-symmetric. However, due to Lemma 3.3.26 and Lemma 3.3.27, it does not satisfy the Jacobi identity:

$$\begin{aligned}
-d\iota(v_\alpha \wedge v_\beta \wedge v_\gamma)\omega &= -\iota([v_\alpha, v_\beta] \wedge v_\gamma)\omega + \iota([v_\alpha, v_\gamma] \wedge v_\beta)\omega - \iota([v_\beta, v_\gamma] \wedge v_\alpha)\omega \\
&= -\{\{\alpha, \beta\}, \gamma\} + \{\{\alpha, \gamma\}, \beta\} - \{\{\beta, \gamma\}, \alpha\} \\
&= \{\alpha, \{\beta, \gamma\}\} + \{\beta, \{\gamma, \alpha\}\} + \{\gamma, \{\alpha, \beta\}\},
\end{aligned}$$

i.e., the Jacobi identity is satisfied only up to an exact term [4].

[4] Note that this exact term vanishes if ω is a symplectic, i.e., 2-form.

Remark 3.3.28. Recall from Lemma 2.3.18, that in symplectic geometry, $\{f,g\} = \omega(v_f, v_g) = \mathcal{L}_{v_f} g$. This is not true for n-plectic manifolds with $n > 1$, since

$$\begin{aligned}\iota_{v_\alpha \wedge v_\beta}\omega &= -\iota_{v_\beta \wedge v_\alpha}\omega \\ &= -\iota_{v_\alpha}\iota_{v_\beta}\omega \\ &= \iota_{v_\alpha} d\beta \\ &= \mathcal{L}_{v_\alpha}\beta - d\iota_{v_\alpha}\beta.\end{aligned}$$

This suggests that we could alternatively define a bracket of two Hamiltonian forms as $\{\alpha,\beta\} := \mathcal{L}_{v_\alpha}\beta$. However, the bracket defined this way is skew-symmetric only up to an exact form: $\{\alpha,\beta\} + \{\beta,\alpha\} = d(\iota_{v_\alpha}\beta + \iota_{v_\beta}\alpha)$. But it satisfies the Leibniz identity: $\{\alpha,\{\beta,\gamma\}\} = \{\{\alpha,\beta\},\gamma\} + \{\beta,\{\alpha,\gamma\}\}$. For more, see [4, §3] and [50, §6].

Thus, there seems to be no Lie algebra of observables associated to a multisymplectic manifold. However, as C. L. Rogers demonstrated in [50], to every n-plectic manifold one can associate a certain Lie n-algebra. We present the relevant theorem without proof:

Theorem 3.3.29. *([50, Thm. 5.2]) Given an n-plectic manifold, there is a corresponding Lie n-algebra $(L, \{[\ ,\ \dots\ ,\ ]_k\})$ with underlying graded vector space*

$$L^i = \begin{cases} \Omega^{n-1}_{Ham}(M) & i = 0 \\ \Omega^{n-1+i}(M) & 1-n \leq i < 0 \end{cases}$$

and maps $\{[\ ,\ \dots\ ,\ ]_k : L^{\otimes k} \to L \mid 1 \leq k < \infty\}$ defined as

$$[\alpha]_1 = d\alpha, \quad \text{if } |\alpha| < 0$$

and, for $k > 1$,

$$[\alpha_1,\ \dots\ ,\alpha_k]_k = \begin{cases} \zeta(k)\iota(v_{\alpha_1} \wedge \dots \wedge v_{\alpha_k})\omega & \text{if } |\alpha_1 \otimes \dots \otimes \alpha_k| = 0 \\ 0 & \text{if } |\alpha_1 \otimes \dots \otimes \alpha_k| < 0, \end{cases}$$

where v_{α_i} is the Hamiltonian vector field associated to $\alpha_i \in \Omega^{n-1}_{Ham}(M)$, $\zeta(k) = -(-1)^{\frac{k(k+1)}{2}}$, and $\iota(\dots)$ denotes contraction with a multivector field: $\iota(v_{\alpha_1} \wedge \dots \wedge v_{\alpha_k})\omega = \iota_{v_{\alpha_k}} \dots \iota_{v_{\alpha_1}}\omega$.

Definition 3.3.30. The Lie n-algebra defined in Theorem 3.3.29 is called the *Lie n-algebra of observables* corresponding to an n-plectic manifold and denoted $L_\infty(M,\omega)$.

The underlying (co)chain complex of this Lie n-algebra is

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(M) \xrightarrow{d} \Omega^{n-1}_{Ham},$$

with Ω^{n-1}_{Ham} in degree 0 and $C^\infty(M)$ in degree $1-n$.

Example 3.3.31. When $n = 1$, $L_\infty(M, \omega)$ is the Lie algebra of observables $(C^\infty(M), \{\ ,\ \})$ of a symplectic manifold: the underlying vector space is $C^\infty(M)$, and the only nonzero multibracket is given by $[f, g] = \{f, g\} = \omega(v_f, v_g)$.

Example 3.3.32. For a 2-plectic manifold (M, ω), $L_\infty(M, \omega)$ has

$$C^\infty(M) \xrightarrow{d} \Omega^1_{Ham}(M)$$

as the underlying cochain complex. The multibrackets $[\ ,\ ]$, $[\ ,\ ,\ ]$ are given by

$$[\alpha_1, \alpha_2] = \iota(v_{\alpha_1} \wedge v_{\alpha_2})\omega$$

$$[\alpha_1, \alpha_2, \alpha_3] = \omega(v_{\alpha_1}, v_{\alpha_2}, v_{\alpha_3}).$$

3.4 Moment maps in multisymplectic geometry

In this section we introduce the concept of moment map in symplectic geometry, give examples of such moment maps and investigate existence and obstructions to existence of moment maps. The material of this section is based on [7], [18], [53].

3.4.1 Introduction

Now that we have introduced multisymplectic manifolds, a natural question to ask is how to generalize the concept of moment map (2.4) to the multisymplectic setting.

There have been multiple attempts at tackling this problem in the literature. In [21, §4C] a moment map is defined as follows: Let G be a Lie group acting on an n-plectic manifold (M, ω) so that the induced infinitesimal action of $\mathfrak{g}$ is by multisymplectic vector fields. A map

$$J : M \to \mathfrak{g}^* \otimes \wedge^{n-1} T^* M$$

that satisfies

$$d\langle J, x\rangle = -\iota_{v_x}\omega$$

is called a *covariant momentum map* or a *multimomentum map* for this action. We can consider the "dual" map $f : \mathfrak{g} \to \Omega^{n-1}(M)$. This map is clearly a generalization of the weak symplectic moment map: namely, for $n = 1$ we get a map $J : M \to \mathfrak{g}^*$ such that $d\langle J, x\rangle = -\iota_{v_x}\omega$, which is precisely the definition of a weak moment map in symplectic geometry. In [10] further parallels between this map and the symplectic moment map are drawn.

Another generalization of the symplectic moment map is presented in [42]:

Let G be a Lie group acting on an n-plectic manifold (M, ω) preserving ω. Let $P_{n,\mathfrak{g}}$ be the n-th Lie kernel of $\mathfrak{g}$ defined in Definition 2.2.36. An equivariant map

$$\mu : M \to P^*_{n,\mathfrak{g}}$$

such that

$$d\langle \mu(m), p\rangle = -\iota_{v_p}\omega$$

for all $p \in P_{n,\mathfrak{g}}$ is called a *multimoment map* for the action of G on (M, ω). Here v_p is the fundamental multivector field corresponding to p, defined in Definition 2.2.20

For applications of these two notions of moment maps we refer the reader to [21], [10], and [42].

In Chapter 4 we will present yet another generalization of the symplectic moment map to n-plectic geometry.

In this section we present the notion of a moment map introduced by M. Callies, Y. Fregier, C. L. Rogers, and M. Zambon in [7]. This notion generalizes the symplectic moment map and, in a certain sense, the two moment maps described above. Namely, for a Lie algebra action $\mathfrak{g} \to \mathfrak{X}(M)$ on an n-plectic manifold (M, ω), a *homotopy moment map* is defined to be an L_∞-morphism $(f_k) : \mathfrak{g} \to L_\infty(M, \omega)$ such that $-i_{v_x}\omega = d(f_1(x))$ for all $x \in \mathfrak{g}$. When $n = 1$, this definition becomes the definition of symplectic comoment map (Definition 2.4.6). Moreover, the first component of the homotopy moment map is the dual of a covariant momentum map, and the n-th component of an equivariant homotopy moment map restricted to the n-th Lie kernel, is, up to sign, the dual of a multimoment map introduced above.

Examples and applications of this map are extensively studied in [7]. For example, it is shown that, as is the case for the symplectic moment map, a homotopy moment map corresponds to an $(n+1)$-cocycle in the equivariant de Rham complex $C_G(M) := (S(\mathfrak{g}^*) \otimes \Omega(M))^G$, with the differential given by $d_G(\alpha)(x) = d(\alpha(x)) - \iota_{v_x}(\alpha(x))$ for all $x \in \mathfrak{g}$.

3.4.2 Definition and examples

The results of the previous section suggest that we can define a multisymplectic co-moment map as an L_∞-morphism between $\mathfrak{g}$ and $L_\infty(M,\omega)$.

Definition 3.4.1. ([7, Def. 5.1]) Let $\mathfrak{g} \to \mathfrak{X}(M), x \mapsto v_x$ be a Lie algebra action on an n-plectic manifold (M,ω) by Hamiltonian vector fields. A *homotopy moment map* for this action (or a $\mathfrak{g}$ *moment map* for short) is an L_∞-morphism

$$\{f_k\} : \mathfrak{g} \to L_\infty(M,\omega)$$

such that

$$-\iota_{v_x}\omega = d(f_1(x)) \qquad \text{for all } x \in \mathfrak{g}.$$

In other words, a homotopy moment map $\{f_k\} : \mathfrak{g} \to L_\infty(M,\omega)$ is a lift of the Lie algebra morphism $\mathfrak{g} \to \mathfrak{X}_{Ham}(M), x \to v_x$ in the category of L_∞-algebras. This can be illustrated by the diagram below, where the horizontal map is the Lie algebra action, and the vertical map is the L_∞-morphism that in degree 0 assigns to a Hamiltonian $(n-1)$-form its Hamiltonian vector field, and is zero in all other degrees. This map, which we denote by h, is a strict L_∞-morphism. Indeed, since the only non-vanishing component of the map is in degree zero, we only have to check that

$$h([\alpha_1, \alpha_2]_2) = [h(\alpha_1), h(\alpha_2)],$$

where the bracket on the left is the multibracket $[\ ,\ ]_2$ defined in Theorem 3.3.29, and the bracket on the right is the Lie bracket of Hamiltonian vector fields. This identity holds by definition of the bracket $[\ ,\ ]_2$ and Lemma 3.3.27.

$$\begin{array}{ccc} & & L_\infty(M,\omega) \\ & \overset{\{f_k\}}{\nearrow} & \big\downarrow h \\ \mathfrak{g} & \longrightarrow & \mathfrak{X}_{Ham}(M) \end{array}$$

Using Theorem 3.3.29 and Definition 3.2.20, we can conclude (see [7, §5]) that a homotopy moment map $\mathfrak{g} \to L_\infty(M,\omega)$ consists of a collection of n graded skew-symmetric maps

$$f_k : \mathfrak{g}^{\otimes k} \to L, \ \ 1 \leq k \leq n$$

of degree $|f_k| = 1 - k$, such that

$$-\iota_{v_x}\omega = d(f_1(x)) \tag{3.14}$$

and the following equations hold:

$$\sum_{1\leq i<j\leq k}(-1)^{i+j+1}f_{k-1}([x_i,x_j],x_1,...,\widehat{x_i},...,\widehat{x_j},...,x_k)=df_k(x_1,...,x_k)$$
$$+\zeta(k)\iota(v_{x_1}\wedge...\wedge v_{x_k})\omega \tag{3.15}$$

for $2\leq k\leq n$ and

$$\sum_{1\leq i<j\leq n+1}(-1)^{i+j+1}f_n([x_i,x_j],x_1,...,\widehat{x_i},...,\widehat{x_j},...,x_{n+1})=\zeta(n+1)\iota(v_{x_1}\wedge...\wedge v_{x_{n+1}})\omega, \tag{3.16}$$

where v_{x_i} is the vector field associated to x_i via the $\mathfrak{g}$-action, and $\zeta(k)=-(-1)^{\frac{k(k+1)}{2}}$.

Note that these equations can be written in a shorter way; namely, for $1\leq k\leq n+1$ and all $p\in\wedge^k\mathfrak{g}$:

$$-f_{k-1}(\delta_k(p))=df_k(p)+\zeta(k)\iota_{v_p}\omega \tag{3.17}$$

where δ_k is the k-th Lie algebra homology differential (Definition 2.2.35), v_p is the fundamental vector field corresponding to p (Definition 2.2.20), and f_0 and f_{n+1} are defined to be zero: $f_0=f_{n+1}=0$.

Remark 3.4.2. For the algebraically inclined reader, we offer yet another (equivalent, but different in flavor) definition of homotopy moment maps used in [14] and [13].

Let M be an n-plectic manifold with a multisymplectic action of a Lie group G. Let $(S^\bullet(s\mathfrak{g}),\tilde{l})$ be the differential graded co-commutative co-algebra, as defined in §3.2.2, corresponding to the Lie algebra $\mathfrak{g}$, and let $\Omega^\bullet_{tr(n)}(M)$ be the de Rham complex of M truncated at n and degree-shifted by $n+1$, i.e.,

$$\Omega^\bullet_{tr(n)}(M):=C^\infty(M)\oplus\cdots\oplus\Omega^{n-1}(M)\oplus\Omega^n(M),$$

where $C^\infty(M)$ has degree $-n-1$, $\Omega^1(M)$ has degree $-n$, $\Omega^2(M)$ has degree $-n+1$, etc., and, finally, $\Omega^n(M)$ has degree -1.

Define h to be the map $h:S^\bullet(s\mathfrak{g})\to\Omega^\bullet(M),p\mapsto-\zeta(k)\iota_{v_p}\omega$, where v_p is the fundamental vector field corresponding to p (Definition 2.2.20). By Lemma 3.3.26, this is a cochain map. Then a homotopy moment map $f:\mathfrak{g}\to L_\infty(M,\omega)$ is a cochain homotopy between the cochain map $h:S^\bullet(s\mathfrak{g})\to\Omega^\bullet(M)$ and the zero map:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & S^{n+1}(s\mathfrak{g}) & \xrightarrow{\tilde{l}} & S^{n}(s\mathfrak{g}) & \xrightarrow{\tilde{l}} & \cdots & \xrightarrow{\tilde{l}} & S^{2}(s\mathfrak{g}) & \xrightarrow{\tilde{l}} & S^{1}(s\mathfrak{g}) & \xrightarrow{\tilde{l}} & 0 \\
 & & \downarrow h & \swarrow f & \downarrow h & \swarrow f & & \swarrow f & \downarrow h & \swarrow f & \downarrow h & & \\
0 & \longrightarrow & C^{\infty}(M) & \xrightarrow{d} & \Omega^{1}(M) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{n-1}(M) & \xrightarrow{d} & \Omega^{n}(M) & \longrightarrow & 0
\end{array}$$

We will not use this definition in this thesis, but it certainly has its merits.

Now that we have defined homotopy moment maps, we present some examples:

Example 3.4.3. (**Symplectic moment maps**) When $n = 1$, a homotopy moment map is the already familiar moment map, so all examples of moment maps from Chapter 2 are examples of homotopy moment maps.

The following example generalizes Example 2.4.12.

Example 3.4.4. (**Exact n-plectic forms**) ([7, Ex. 8.1]) Let G be a Lie group acting on an n-plectic manifold (M, ω). Assume $\omega = d\alpha$, and that α is G-invariant. Then $\{f_k\} : \mathfrak{g} \to L_\infty(M, \omega)$ given by

$$f_k : \mathfrak{g}^{\otimes k} \to \Omega^{n-k}(M) \;\; 1 \leq k \leq n$$

$$f_k(x_1, ..., x_k) = (-1)^{k-1}\zeta(k)\iota(v_{x_1} \wedge \cdots \wedge v_{x_k})\alpha \tag{3.18}$$

is a homotopy moment map for the action of G.

A special case of the above is the following example:

Example 3.4.5. ([7, Ex. 8.4]) Consider $SO(n)$ acting on the $(n-1)$-plectic manifold $(\mathbb{R}^n, \omega)$ where $\omega = dx_1 \wedge dx_2 \wedge ... \wedge dx_n$ is a volume form. Then the following form is an $SO(n)$-invariant primitive of ω:

$$\alpha = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}x_k dx_1 ... \widehat{dx_k} ... dx_n.$$

Let $\{e_{ij} : 1 \leq i < j \leq n\}$ be the basis of $\mathfrak{so}(n)$, where e_{ij} is the matrix that has -1 on position (i, j), 1 on position (j, i), and zeroes everywhere else. The corresponding fundamental vector fields are then given by

$$v_{e_{ij}} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j},$$

and the moment map can be calculated by the formulae (3.18) on the basis elements and extended multilinearly to all of $\wedge^k \mathfrak{g}$ for $1 \leq k \leq n$.

3.4.2.1 Multisymplectic coadjoint orbits

We consider the 2-plectic manifolds introduced in §3.3.1.1, with the cocycle c given by $c = d_{\mathfrak{g}}b$, where $b \in \wedge^2\mathfrak{g}^* \otimes \mathfrak{h}$ is an invariant 2-form in the sense of Definition 3.3.9.

Namely, consider ρ a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ of a Lie algebra $\mathfrak{g}$, the induced representation of $\mathfrak{g}$ on the dual $\mathfrak{h}^*$, and the action of G, the simply-connected Lie group integrating $\mathfrak{g}$, on $\mathfrak{h}^*$. Consider a 3-form on an orbit σ_ξ of $\xi \in \mathfrak{h}^*$ under the latter action, given by

$$\omega_\xi(v_x, v_y, v_z) := \xi(d_{\mathfrak{g}}b(x, y, z)), \tag{3.19}$$

where the v_x, v_y, v_z are the fundamental vector fields corresponding to x, y, z respectively.

Let $b \in \wedge^2\mathfrak{g}^* \otimes \mathfrak{h}$ be invariant, and assume that it satisfies

$$\mathfrak{g}_\xi \subset ker(\xi \circ b) \tag{3.20}$$

at ξ.

The next lemma shows that if the condition (3.20) is satisfied at ξ, then it is satisfied at all points of σ_ξ.

Lemma 3.4.6. *If $\mathfrak{g}_\xi \subseteq ker(\xi \circ b)$, then $\mathfrak{g}_{g\xi} \subseteq ker(g\xi \circ b)$ for all $g \in G$*

Proof. Analogously to the proof of Proposition 3.3.10, note that $\mathfrak{g}_{g\xi} = Ad_g\mathfrak{g}_\xi$. Suppose $x \in ker(\xi \circ b)$, we need to show that $Ad_g x \in ker(g\xi \circ b)$. For all $y \in \mathfrak{g}$, we have $(g\xi \circ b)(Ad_g x, y) = \xi(g^{-1}(b(Ad_g x, y))) = \xi(b(Ad_{g^{-1}}Ad_g x, Ad_{g^{-1}}y))$, by the invariance of b. □

This condition will be needed for certain forms to be well-defined. The next lemma shows that if this condition is satisfied for an invariant b, then it is also satisfied for the 3-cocycle $d_{\mathfrak{g}}b$ obtained from b. Moreover, this 3-cocycle is also invariant.

Lemma 3.4.7. *Let $b \in \wedge^2\mathfrak{g}^* \otimes \mathfrak{h}$ be invariant, and assume that it satisfies*

$$\mathfrak{g}_{g\xi} \subset ker(g\xi \circ b) \tag{3.21}$$

at $g\xi$. Then $d_{\mathfrak{g}}b$ satisfies condition (3.9) *at $g\xi$. Furthermore, $d_{\mathfrak{g}}b$ is invariant.*

Proof. We show that if $\mathfrak{g}_{g\xi} \in \ker(g\xi \circ b)$, then $\mathfrak{g}_{g\xi} \in \ker(g\xi \circ d_{\mathfrak{g}}b)$. Let $x \in \mathfrak{g}_{g\xi} \cap \ker(g\xi \circ b)$. Writing out $d_{\mathfrak{g}}b(x, y, x)$ we get

$$(d_{\mathfrak{g}}b)(x, y, z) = \rho(x)b(y, z) - \rho(y)b(x, z) + \rho(z)b(x, y)$$
$$- b([x, y], z) + b([x, z], y) - b([y, z], x).$$

Using the invariance of b, we have $\rho(x)b(y, z) = b([x, y], z) + b(y, [x, z])$, and similarly for the other two terms in the first line. Hence,

$$(d_{\mathfrak{g}}b)(x, y, z) = b([x, y], z) - b([x, z], y) + b([y, z], x) \tag{3.22}$$

or equivalently, using the invariance of b again,

$$(d_{\mathfrak{g}}b)(x, y, z) = \rho(x)b(y, z) - b([z, y], x) \tag{3.23}$$

Using equation 3.23, we obtain

$$(g\xi \circ d_{\mathfrak{g}}b)(x, y, z) = g\xi(\rho(x)b(y, z)) - g\xi(b([z, y], x)) = 0,$$

because $x \in \mathfrak{g}_{g\xi}$ and $x \in \ker(g\xi \circ \beta)$. Thus, $x \in \ker(g\xi \circ d_{\mathfrak{g}}b)$.

To prove invariance, we have to show

$$\rho(u)((d_{\mathfrak{g}}b)(x, y, z)) = (d_{\mathfrak{g}}b)(ad_u x, y, z) + (d_{\mathfrak{g}}b)(x, ad_u y, z) + (d_{\mathfrak{g}}b)(x, y, ad_u z) \tag{3.24}$$

Using (3.22) and invariance of b, the left-hand side of (3.24) becomes:

$$\rho(u)((d_{\mathfrak{g}}b)(x, y, z)) = \rho(u)(b([x, y], z) - b([x, z], y) + b([y, z], x)) \tag{3.25}$$
$$= b(ad_u[x, y], z) + b([x, y], ad_u z) - b(ad_u[x, z], y)$$
$$- b([x, z], ad_u y) + b(ad_u[y, z], x) + b([y, z], ad_u x).$$

The right-hand side of (3.24), using (3.22), becomes

$$(d_{\mathfrak{g}}b)(ad_u x, y, z) + (d_{\mathfrak{g}}b)(x, ad_u y, z) + (d_{\mathfrak{g}}b)(x, y, ad_u z) = \tag{3.26}$$
$$b([ad_u x, y]z) - b([ad_u x, z], y) + b([y, z], ad_u x)$$
$$+b([x, ad_u y], z) - b([x, z], ad_u y) + b([ad_u y, z], x)$$
$$+b([x, y], ad_u z) - b([x, ad_u z], y) + b([y, ad_u z], x).$$

Subtracting the right-hand side of (3.26) from the right-hand side of (3.25) gives 0, due to bilinearity of b and the Jacobi identity. $\square$

Let $b \in \wedge^2\mathfrak{g}^* \otimes \mathfrak{h}$ be G-invariant and let it satisfy condition (3.20). Consider the expression

$$\beta_\xi(v_x, v_y) := \xi(b(x,y)) \tag{3.27}$$

Then β is a well-defined 2-form on σ_ξ, by Lemma 3.4.6. The next lemma shows that it is also G-invariant, with the proof being completely analogous to the proof of Proposition 3.3.13:

Lemma 3.4.8. *Let $b \in \wedge^2\mathfrak{g}^* \otimes \mathfrak{h}$ be G-invariant, and let β be defined as in* (3.27)*. Then β is invariant under the G-action on σ_ξ.*

Then $c := d_\mathfrak{g} b$ is invariant and satisfies condition (3.9) by Lemma 3.4.7. Thus formula (3.7) defines a closed, invariant 3-form ω on σ_ξ, by Proposition 3.3.11 and Proposition 3.3.13. The following proposition shows that ω is de Rham-exact, and has β as its primitive.

Proposition 3.4.9. *Let $b \in \wedge^2\mathfrak{g}^* \otimes h$ be invariant, and let $c := d_\mathfrak{g} b$. Define a 2-form β on σ_ξ by*

$$\beta_\xi(v_x, v_y) := \xi(b(x,y)).$$

and consider ω defined by

$$\omega_\xi(v_x, v_y, v_x) := (\xi \circ d_\mathfrak{g} b)(x,y,z).$$

Then $d\beta = \omega$.

Proof. The proof is very similar to the proof of Proposition 3.3.11.

$$\begin{aligned}
(d\beta)_\xi(v_x, v_y, v_z) &= (v_x \cdot \beta(v_y, v_z))_\xi - (v_y \cdot \beta(v_x, v_z))_\xi + (v_z \cdot \beta(v_x, v_y))_\xi \\
&\quad - \beta_\xi([v_x, v_y], v_z) + \beta_\xi([v_x, v_z], v_y) - \beta_\xi([v_y, v_z], v_x) = \\
&= \xi(\rho(x) \cdot b(y,z)) - \xi(\rho(y) \cdot b(x,z)) + \xi(\rho(z) \cdot b(x,y)) \\
&\quad - \xi(b([x,y],z)) + \xi(b([x,z],y)) - \xi(b([y,z],x)) = \\
&= (\xi \circ d_\mathfrak{g} b)(x,y,z) \\
&= (\omega_\xi)(v_x, v_y, v_z)
\end{aligned}$$

□

Hence, ω has a G-invariant primitive, and Example 3.4.4 tells us that there exists a homotopy moment map for the action of G on (σ_ξ, ω):

Proposition 3.4.10. *The G-action on (σ_ξ, ω) admits an equivariant homotopy moment map given by*

$$f_1 : \mathfrak{g} \to \Omega^1_{Ham}(\sigma_\xi), x \mapsto \iota_{v_x}\beta$$

$$f_2 : \wedge^2\mathfrak{g} \to C^\infty(\sigma_\xi), x \wedge y \mapsto -\iota_{v_y}\iota_{v_x}\beta$$

We finish by noting that examples 3.3.16, 3.3.17, 3.3.18 satisfy the condition 3.20 for $\beta = Id_{\wedge^2\mathfrak{g}}$, and therefore they all possess a homotopy moment map.

3.4.3 Existence and obstruction

In this subsection, we present results relating existence of homotopy moment maps $\mathfrak{g} \to L_\infty(M, \omega)$ to certain class in Lie algebra cohomology of $\mathfrak{g}$. The exposition in this subsection closely follows [7, §9].

Let (M, ω) be an n-plectic manifold, and let G be a Lie group that acts on (M, ω), and let this action preserve ω. Consider the map $c_p^{\mathfrak{g}} : \wedge^{n+1}\mathfrak{g} \to \mathbb{R}$ given by

$$c_p^{\mathfrak{g}} : \wedge^{n+1}\mathfrak{g} \to \mathbb{R} \tag{3.28}$$

$$x_1 \wedge ... \wedge x_{n+1} \mapsto (-1)^n\zeta(n+1)\iota(v_1 \wedge ... \wedge v_{n+1})\omega|_p,$$

where $p \in M$, and v_i is the fundamental vector field corresponding to x_i, and $\zeta(n+1) = -(-1)^{\frac{(n+1)(n+2)}{2}}$.

Lemma 3.4.11. *For all $p \in M$, the map $c_p^{\mathfrak{g}} : \wedge^{n+1}\mathfrak{g} \to \mathbb{R}$ defined in* (3.28) *is a cocycle in the Chevalley-Eilenberg complex $CE(\mathfrak{g})$ of $\mathfrak{g}$ with values in the trivial representation $\mathbb{R}$.*

Proof. Computing $d_{\mathfrak{g}}c_p^{\mathfrak{g}}$ gives:

$$d_{\mathfrak{g}}c_p^{\mathfrak{g}}(x_1, ..., x_{n+2}) = \sum_{i<j}(-1)^{i+j}c_p^{\mathfrak{g}}([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+2})$$

$$= \zeta(n+1)\sum_{i<j}(-1)^{n+i+j}\iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{n+2})\omega|_p.$$

By Lemma 3.3.26, taking into account that the action preserves ω,

$$\zeta(n+1)\sum_{i<j}(-1)^{n+i+j}\iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{n+2})\omega|_p =$$

$$\zeta(n+1)(-1)^{2n+2}d(\iota(v_1 \wedge \cdots \wedge v_{n+2})\omega)|_p = 0,$$

where the last equality holds because $\omega \in \Omega^{n+1}(M)$.

□

Thus, $c_p^{\mathfrak{g}}$ defines a class $[c_p^{\mathfrak{g}}]$ in Lie algebra cohomology of $\mathfrak{g}$. It can be shown that, if M is connected, this class does not depend on the choice of $p \in M$ (see [7, Cor. 9.3]).

The following proposition shows that this class is an obstruction to the existence of a homotopy moment map:

Proposition 3.4.12. *([7, Prop. 9.5]) Let (M, ω) be a connected n-plectic manifold, and let G be a Lie group acting on (M, ω). If this action admits a homotopy moment map, then $[c_p^{\mathfrak{g}}] = 0$.*

Proof. Let (f_k) be a moment map for this action. Consider a map $b : \wedge^n \mathfrak{g} \to \mathbb{R}$ defined by
$$b(x_1, ..., x_n) = (-1)^{n+1} f_n(x_1, ..., x_n)|_p$$
for a fixed $p \in M$. By (3.16), we have

$$c_p^{\mathfrak{g}}(x_1, ..., x_{n+1}) = \sum_{1 \le i < j \le n+1} (-1)^{i+j} b([x_i, x_j], x_1, ..., \widehat{x_i}, ..., \widehat{x_j}, ..., x_{n+1}),$$

i.e., $c_p^{\mathfrak{g}} = d_{\mathfrak{g}} b$. □

Under some conditions on the de Rham cohomology of M, we also have the converse:

Theorem 3.4.13. *[7, Thm. 9.6] Let (M, ω) be a connected n-plectic manifold, and let G be a Lie group acting on (M, ω) the induced Lie algebra action of which is by Hamiltonian vector fields. Let $\phi : \mathfrak{g} \to \Omega^{n-1}_{Ham}(M)$ be a linear map that satisfies*
$$d(\phi(x)) = -\iota_{v_x} \omega,$$
for all $x \in \mathfrak{g}$. If $[c_p^{\mathfrak{g}}] = 0$ for $c_p^{\mathfrak{g}}$ defined in (3.28), and $H^i(M) = 0$ for $1 \le i \le n-1$, then there exists a homotopy moment map $(f_k) : \mathfrak{g} \to L_\infty(M, \omega)$ such that $f_1 = \phi$.

Proof. We will give a sketch of the proof. For details we refer the reader to the proof of [7, Thm. 9.6]. The first step of the proof is to show that, if, for every

$2 \leq k \leq n+1$, a map $f_{k-1} : \mathfrak{g}^{\otimes(k-1)} \to \Omega^{n-k+1}(M)$ satisfies the equation[5]

$$\sum_{1\leq i<j\leq k-1} (-1)^{i+j+1} f_{k-2}([x_i, x_j], x_1, ..., \widehat{x_i}, ..., \widehat{x_j}, ..., x_{k-1}) = d(f_{k-1}(x_1, ..., x_{k-1})) + \zeta(k-1)\iota(v_{x_1} \wedge ... \wedge v_{x_{k-1}})\omega,$$

then the following expression is a closed $(n+1-k)$-form for all $x_1, ..., x_2 \in \mathfrak{g}$:

$$\sum_{1\leq i<j\leq k} (-1)^{i+j+1} f_{k-1}([x_i, x_j], x_1, ..., \widehat{x_i}, ..., \widehat{x_j}, ..., x_k) - \zeta(k)\iota(v_{x_1} \wedge ... \wedge v_{x_k})\omega.$$

The proof of this fact uses Lemma 3.3.26. Then, using induction on k and the assumption that $H^i(M) = 0$ for $1 \leq i \leq n-1$, it follows that there exist (f_k) satisfying equations (3.15) for $2 \leq k \leq n$.

The f_n constructed this way will not necessarily satisfy (3.16). However, consider

$$h(x_1, ..., x_{n+1}) = -\sum_{1\leq i<j\leq n+1} (-1)^{i+j+1} f_n([x_i, x_j], x_1, ..., \widehat{x_i}, ..., \widehat{x_j}, ..., x_{n+1}) + \zeta(n+1)\iota(v_{x_1} \wedge ... \wedge v_{x_{n+1}})\omega.$$

Fix $p \in M$ and evaluate both summands on the right-hand side of the above expression at p. Note that the first summand then becomes $d_{\mathfrak{g}} f_n|_p$, and the second one is $(-1)^n c_p^{\mathfrak{g}}$, which is $d_{\mathfrak{g}}$-exact by assumption of the theorem. Therefore, $h|_p$ is $d_{\mathfrak{g}}$-exact, i.e., $h|_p = d_{\mathfrak{g}} b$ for some $b \in \wedge^n \mathfrak{g}^*$. However, by the result we established in the beginning of the proof, $h(x_1, ..., x_{n+1})$ is a closed 0-form for all $x_1, ..., x_{n+1} \in \mathfrak{g}$, i.e., a constant function, since M is connected. Therefore, $h = d_{\mathfrak{g}} b \in \wedge^{n+1}\mathfrak{g}^*$. Then, $f_1, ..., f_{n-1}, f_n - b$ are the components of a homotopy moment map. □

We will see in the next section, that the assumption on the cohomology of M can be weakened: only certain components of $H^\bullet_{CE}(\mathfrak{g}) \otimes H^\bullet(M)$ have to vanish for a homotopy moment map to exist.

3.4.3.1 Moment maps and central n-extensions

The following construction is the higher analogue of the one in 2.4.3.3.

Proposition 3.4.14. *[7, Prop. 9.10] Let (M, ω) be a connected n-plectic manifold, and let G be a Lie group acting on (M, ω) the induced Lie algebra*

[5] Note that this is the equation (3.15) for $k-1$.

action of which is by Hamiltonian vector fields $\mathfrak{g} \to \mathfrak{X}_{Ham}(M)$. Assume that $H^i(M) = 0$ for $1 \leq i \leq n-1$. Let $p \in M$, and let $\bar{\mathfrak{g}}_c$ be the central extension of $\mathfrak{g}$ corresponding to the cocycle $c_p^{\mathfrak{g}}$ defined in (3.28)*. There exists an L_∞-morphism*

$$\bar{f} : \bar{\mathfrak{g}}_c \to L_\infty(M, \omega)$$

such that for all $x \in \mathfrak{g}$

$$d(\bar{f}_1(x)) = -\iota_{v_x}\omega,$$

where v_x is the fundamental vector field associated to x.

Proof. Using Definition 3.2.20 and Theorem 3.3.29, we can see that ([7, Prop. A.9]) an L_∞-morphism $\bar{f} : \bar{\mathfrak{g}}_c \to L_\infty(M)$ is a collection of n skew-symmetric maps of degree $1 - k$

$$\bar{f}_1 : \mathfrak{g} \to \Omega^{n-1}_{Ham}(M),\ \bar{f}_1 : \mathbb{R} \to C^\infty(M)$$

$$\bar{f}_k : \mathfrak{g}^{\otimes k} \to L_\infty(M, \omega)\ \ 2 \leq k \leq n$$

satisfying the following equations:

$$d(\bar{f}_1(r)) = 0 \text{ for all } r \in \mathbb{R}$$

$$\sum_{1 \leq i < j \leq k} (-1)^{i+j+1} \bar{f}_{k-1}([x_i, x_j], x_1, ..., \widehat{x_i}, ..., \widehat{x_j}, ..., x_k) = d(\bar{f}_k(x_1, ..., x_k)) + \zeta(k)\iota(v_{x_1} \wedge ... \wedge v_{x_k})\omega \tag{3.29}$$

for $2 \leq k \leq n$ and

$$\sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} \bar{f}_n([x_i, x_j], x_1, ..., \widehat{x_i}, ..., \widehat{x_j}, ..., x_{n+1}) + \bar{f}_1(c(x_1, ..., x_{n+1})) = \zeta(n+1)\iota(v_{x_1} \wedge ... \wedge v_{x_{n+1}})\omega, \tag{3.30}$$

for all $x_i \in \mathfrak{g}$.

We define $\bar{f}_1$ by $\bar{f}_1(x) = \alpha$, $\bar{f}_1(r) = (-1)^n r$ for $x \in \mathfrak{g}$, $r \in \mathbb{R}$, and where $\alpha \in \Omega^{n-1}_{Ham}$ is a Hamiltonian $n-1$ form whose Hamiltonian vector field is v_x.

We proceed further as in the proof[6] of Theorem 3.4.13, and obtain maps $\bar{f}_k$ for $2 \leq k \leq n$ that satisfy equation (3.29).

Finally, consider the map $b : \wedge^n \mathfrak{g} \to \mathbb{R}$ defined by

$$b(x_1, ..., x_n) := \bar{f}_n(x_1, ..., x_n)|_p,$$

and define

$$\bar{f}'_n := \bar{f}_n - b.$$

It is clear that $\bar{f}'_n$ satisfies the equation (3.29), since $\underline{b(x_1, ..., x_n)}$ is a constant function for all $x_i \in \mathfrak{g}$. Using Lemma 3.3.26 and the fact that $\bar{f}'_n$ satisfies the equation (3.29), i.e.,

$$\sum_{1 \leq i < j \leq k} (-1)^{i+j+1} \bar{f}_{n-1}([x_i, x_j], x_1, ..., \widehat{x_i}, ..., \widehat{x_j}, ..., x_n) = d(\bar{f}'_n(x_1, ..., x_n)) + \zeta(n)\iota(v_{x_1} \wedge ... \wedge v_{x_n})\omega$$

we conclude that the function

$$F := \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} \bar{f}'_n([x_i, x_j], x_1, ..., \widehat{x_i}, ..., \widehat{x_j}, ..., x_{n+1}) + (-1)^n c_p^{\mathfrak{g}}(x_1, ..., x_{n+1}) - \zeta(n+1)\iota(v_{x_1} \wedge ... \wedge v_{x_{n+1}})\omega,$$

is closed for all $x_i \in \mathfrak{g}$. Therefore,

$$F = F(p) = (-1)^n c_p^{\mathfrak{g}}(x_1, ..., x_{n+1}) - \zeta(n+1)\iota(v_{x_1} \wedge ... \wedge v_{x_{n+1}})\omega|_p = 0.$$

Thus, $\bar{f}'$ satisfies equation (3.30), and $\bar{f} = (\bar{f}_1, ..., \bar{f}')$ is the required morphism. □

Example 3.4.15. ([7, Ex. 9.11]) (**Heisenberg n-algebra**) We saw in Example 2.2.40 that the Heisenberg Lie algebra $\mathfrak{h}$ with the following commutator relations between the basis elements

$$[x, y] = z, [x, z] = 0, [y, z] = 0$$

is a central extension $\mathfrak{h} = \mathbb{R}^2 \oplus \mathbb{R}$ of the abelian Lie algebra $\mathbb{R}^2$ corresponding to the cocycle given by the canonical symplectic form ω on $\mathbb{R}^2$. We also saw

[6] Note the similarity between the equations (3.29) and the equations (3.15) for homotopy moment maps.

in Example 2.4.22 that the action of $\mathbb{R}^2$ on itself by translations does not admit a moment map. However, by results of §2.4.3.3, there is a Lie algebra homomorphism $\bar{\mu}^* : \mathfrak{h} \to C^\infty(M)$ such that $d(\bar{\mu}^*(x,r)) = -\iota_{v_x}\omega$ for all $x \in \mathbb{R}^2$, i.e., a comoment map for the action of $\mathfrak{h}$.

The following example generalizes this to arbitrary vector spaces. Namely, let V be a real finite-dimensional vector space, and let $\omega \in \wedge^{n+1}V^*$ be a nonzero multilinear $n+1$-form on V. If we consider V as a manifold, ω induces a differential $n+1$-form by translation along V, which we also denote by ω. Such a form is then closed and invariant under translations of V by itself considered as an abelian Lie group. Since V is a vector space, and therefore is simply-connected, the induced action of $\mathfrak{g} = V$ is by Hamiltonian vector fields. However, since V is abelian, the co-cycle $c_p^{\mathfrak{g}}$ defined by ω can only be exact if it vanishes. Since we chose ω to be nonzero, the class $[c_p^{\mathfrak{g}}]$ corresponding to ω does not vanish, and the action does not admit a homotopy moment map.

However, by Proposition 3.4.14, there is an L_∞-morphism $\{\bar{f}_k\}$ between the central n-extension $\bar{V}_c$ of V corresponding to the co-cycle $c_p^{\mathfrak{g}}$ and $L_\infty(M,\omega)$, such that $d(\bar{f}_1(x)) = -\iota_{v_x}\omega$ for all $x \in V$.

Example 3.4.16. ([7, 9.12]) Let G be a connected compact simple group, and $\omega = \langle \theta^L, [\theta^L, \theta^L] \rangle$ as in Example 3.3.8. Then ω is invariant under (left) translations of G, but the class $[c_p^{\mathfrak{g}}]$ does not vanish. The central 2-extension of $\mathfrak{g}$ corresponding to $c_p^{\mathfrak{g}}$ is called the *string Lie 2-algebra* and denoted $\mathfrak{string}(\mathfrak{g})$. By Proposition 3.4.14, there is an L_∞-morphism $\{\bar{f}_k\}$, $1 \le k \le 3$, between $\mathfrak{string}(\mathfrak{g})$ and $L_\infty(M,\omega)$, such that $d(\bar{f}_1(x)) = -\iota_{v_x}\omega$ for all $x \in \mathfrak{g}$.

3.4.4 Characterization of homotopy moment maps in terms of a double complex

In this subsection it will be demonstrated that homotopy moment maps $\mathfrak{g} \to L_\infty(M)$ correspond to primitives of a certain element in a certain double complex that combines the Lie algebra cohomology of $\mathfrak{g}$ and the de Rham cohomology of M. This subsection closely follows [18] and [53].

Let G be a Lie group acting on an n-plectic manifold (M,ω) by preserving ω, and let $\mathfrak{g}$ be its Lie algebra. Consider the double complex

$$(\wedge^{\ge 1}\mathfrak{g}^* \otimes \Omega(M), d_{\mathfrak{g}}, d), \tag{3.31}$$

where $d_{\mathfrak{g}}$ is the Chevalley-Eilenberg differential of $\mathfrak{g}$ and d is the de Rham differential of M. We will denote the total complex of this double complex by (C, d_{tot}), where

$$d_{tot} := d_{\mathfrak{g}} \otimes 1 + 1 \otimes d. \tag{3.32}$$

Since we are using the Koszul sign convention, for $f \in \wedge^k \mathfrak{g}^* \otimes \Omega^l(M)$ we have

$$d_{tot} f = d_{\mathfrak{g}} f + (-1)^k df,$$

with $d_{\mathfrak{g}} f \in \wedge^{k+1}\mathfrak{g}^* \otimes \Omega^l(M)$ and $df \in \wedge^k \mathfrak{g}^* \otimes \Omega^{l+1}(M)$ evaluated on elements of $\wedge^{\geq 1}\mathfrak{g}^*$ in the following way:

$$d_{\mathfrak{g}} f(x_1, ..., x_{k+1}) = f(\delta_{k+1}(x_1, ..., x_{k+1}))$$
$$df(x_1, ..., x_k) = d(f(x_1, ..., x_k)),$$

where δ_{k+1} is the $(k+1)$-st Lie algebra homology differential (Definition 2.2.35).

Define $\omega_k : \wedge^k \mathfrak{g} \to \Omega^{n+1-k}$ by

$$\omega_k : \wedge^k \mathfrak{g} \to \Omega^{n+1-k}$$
$$(x_1, ..., x_k) \mapsto \iota(v_{x_1} \wedge ... \wedge v_{x_k})\omega, \tag{3.33}$$

where v_{x_i} is the fundamental vector field corresponding to $x_i \in \mathfrak{g}$.

Consider an element $\widetilde{\omega} \in C$ defined by

$$\widetilde{\omega} := \sum_{k=1}^{n+1} (-1)^{k-1} \omega_k. \tag{3.34}$$

We then have the following:

Lemma 3.4.17. *$\widetilde{\omega}$ is closed, i.e., $d_{tot}\widetilde{\omega} = 0$.*

Proof. $d_{tot}\omega = \sum_{k=1}^{n+1} (-1)^{k-1}(d_{\mathfrak{g}} + (-1)^k d)\omega_k$. By Lemma 3.3.26,

$$(-1)^k d\omega_k = d_{\mathfrak{g}} \omega_{k-1},$$

or equivalently,

$$(-1)^{k-1} d_{\mathfrak{g}} \omega_k = (-1)^{k-1}(-1)^{k+1} d\omega_{k+1} = d\omega_{k+1}.$$

Thus,

$$\begin{aligned} d_{tot}\omega &= \sum_{k=1}^{n+1}(-1)^{k-1}(d_{\mathfrak{g}} + (-1)^k d)\omega_k = \sum_{k=1}^{n+1} d\omega_{k+1} + (-1)^{2k-1} d\omega_k \\ &= \sum_{k=1}^{n+1} d\omega_{k+1} - d\omega_k = d\omega_{n+2} - d\omega_1 \\ &= 0, \end{aligned}$$

since $\omega_{n+2} = 0$, because $\omega \in \Omega^{n+1}(M)$, and $(d\omega_1)(x) = d(\omega_1(x))$=$d\iota_{v_x}\omega = 0$, since v_x are multisymplectic vector fields (see Lemma 3.3.21). □

Note that if $\{f_k\} : \mathfrak{g} \to L_\infty(M)$ is a moment map, then its components are graded skew-symmetric maps $f_k : \mathfrak{g}^{\otimes k} \to \Omega^{n-k}(M)$, i.e., $f_k \in \wedge^k \mathfrak{g}^* \otimes \Omega^{n-k}(M)$ so homotopy moment maps are elements of C. It turns out that they correspond to primitives of $\widetilde{\omega}$, as we will see below.

3.4.4.1 Existence and uniqueness

The following proposition establishes a bijective correspondence between homotopy moment maps for the action of G on (M, ω) and primitives of $\widetilde{\omega}$ in (C, d_{tot}).

Proposition 3.4.18. *([18, Prop. 2.5]) Let $\phi = \phi_1 + ... + \phi_n$, where $\phi_k \in \wedge^k \mathfrak{g}^* \otimes \Omega^{n-k}(M)$. Then $d_{tot}\phi = \widetilde{\omega}$ if and only if*

$$f_k := \zeta(k)\phi_k : \wedge^k \mathfrak{g} \to \Omega^{n-k}(M),$$

for $k = 1, ..., n$ are the components of a homotopy moment map for the action of G on (M, ω).

Proof. Writing down the components of $d_{tot}\phi$ and comparing them with the components of $\widetilde{\omega}$ that live in the same space, we get that $d_{tot}\phi = \widetilde{\omega}$ if and only if:

$$\begin{aligned} -d\phi_1 &= \omega_1 \\ d_{\mathfrak{g}}\phi_{k-1} + (-1)^k d\phi_k &= (-1)^{k-1}\omega_k \quad \forall 2 \le k \le n \\ d_{\mathfrak{g}}\phi_n &= (-1)^n \omega_{n+1}. \end{aligned}$$

Evaluating the equations above on $x_1, ..., x_i \in \mathfrak{g}$ for $i = 1, ..., n+1$, we obtain for $2 \leq k \leq n$

$$d(\phi_1(x)) = -\iota_{v_x}\omega$$

$$\sum_{1\leq i<j\leq k} (-1)^{i+j}\phi_{k-1}([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_k) = -(-1)^k d(\phi_k(v_{x_1}, ..., v_{x_k})) + (-1)^{k-1}\iota(v_{x_1} \wedge ... \wedge v_{x_k})\omega$$

$$\sum_{1\leq i<j\leq n+1} (-1)^{i+j}\phi_n([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{n+1}) = (-1)^n\iota(v_{x_1} \wedge ... \wedge v_{x_{n+1}})\omega$$

The first of the equations above is the first equation in the definition of the homotopy moment map. Multiplying the second equation above by $-\zeta(k-1) = -(-1)^k\zeta(k)$ gives equation (3.15), and multiplying the third of the above equations by $-\zeta(n) = -(-1)^{n+1}\zeta(n+1)$ gives equation (3.16).

□

Remark 3.4.19. Note that it follows from Proposition 3.4.18 that the set of homotopy moment maps $\mathfrak{g} \to L_\infty(M, \omega)$ is an affine space.

Thus, by Proposition (3.4.18), homotopy moment map for the given action of $\mathfrak{g}$ on M exists if and only if $\widetilde{\omega}$ is d_{tot}-exact. It also follows from this that if f is a homotopy moment map for the given action, f' is another homotopy moment map for the same action if and only if $f' = f + \xi$, where $\xi \in C$ is such that $d_{tot}\xi = 0$, which answers the question of uniqueness:

Corollary 3.4.20. *Two moment maps for an action of G on (M, ω) differ by a closed element of (C, d_{tot}).*

The following example shows that applying the above results to symplectic manifolds recovers the familiar results from symplectic geometry:

Example 3.4.21. Let $n = 1$, i.e., consider the case of symplectic geometry. Consider the action of a Lie group G on a symplectic manifold (M, ω). Then, by Proposition 3.4.18, we get that $f : \mathfrak{g} \to C^\infty(M)$ is a homotopy moment map for this action if and only if $d_{tot}f = \widetilde{\omega}$, which translates to

$$-df = \omega_1$$

$$d_{\mathfrak{g}}f = -\omega_2$$

or, evaluating on $x, y \in \mathfrak{g}$,

$$d(f(x)) = -\iota_{v_x}\omega$$

$$f([x,y]) = \omega(v_x, v_y),$$

which are precisely the conditions in the definition 2.4.5 of a symplectic moment map.

Two such homotopy moment maps differ by an element $\xi : \mathfrak{g} \to C^\infty(M)$ such that $d_{tot}\xi = 0$, i.e.,

$$d\xi = 0$$

$$d_{\mathfrak{g}}\xi = 0.$$

If M is a connected manifold, then the first equation above means that $\xi \in \mathfrak{g}^*$, and the second equation means that $\xi \in [\mathfrak{g}, \mathfrak{g}]^\circ$, i.e., we obtain the statement of Proposition 2.4.14.

Chapter 4

Weak homotopy moment maps vs. homotopy moment maps

In this chapter we compare two notions of moment maps in n-plectic geometry. The material of this chapter is based on a preprint ([43]) co-authored by the author of the thesis, with similar wording in many places.

4.1 Introduction

In §3.4.1 we mentioned a few generalizations of the symplectic moment map to multisymplectic geometry. One of them was the multi-moment map of Madsen and Swann (see [41] and [42]) defined as an equivariant map $f : M \to P_{n,\mathfrak{g}}^*$ such that $df^*(p) = \iota_{v_p}\omega$, where (M, ω) is an n-plectic manifold equipped with an ω-preserving action of a Lie group G, v_p is the fundamental vector field corresponding to $p \in P_{n,\mathfrak{g}}$, and $f^* : P_{n,\mathfrak{g}} \to C^\infty(M)$ is the map defined by $f^*(p)|_m = f(m)(p)$.

In this section we turn our attention to a generalization of the multimoment map[1] introduced by J. Herman in [30] and [29]. In [30] this map is used to generalize Noether's theorem (2.4.1) to n-plectic geometry.

[1] Or, rather, its dual.

In particular, it was shown in [55] that components of a homotopy moment map restricted to the Lie kernel give rise to conserved quantities, just like in symplectic geometry (c.f. Theorem 2.4.1). Furthermore, the equation (3.16) was not used in the derivation of these results. This suggests considering a map defined in the following way:

Definition 4.1.1. Let $\mathfrak{g} \to \mathfrak{X}(M), x \mapsto v_x$ be a Lie algebra action on an n-plectic manifold (M, ω) by Hamiltonian vector fields. A *weak (homotopy) moment map* is a collection of linear maps $\widehat{f_k} : P_{k,\mathfrak{g}} \to \Omega^{n-k}(M)$, where $1 \leq k \leq n$, satisfying

$$d(\widehat{f_k}(p)) = -\zeta(k)\iota_{v_p}\omega$$

for $k \in 1, ..., n$ and all $p \in P_{k,\mathfrak{g}}$, where $P_{k,\mathfrak{g}}$ is the k-th Lie kernel of $\mathfrak{g}$, and $\zeta(k)$ is as defined in (3.1).

Recall that the Lie kernel was defined in 2.2.36.

Remark 4.1.2. Looking at equations (3.17) for homotopy moment maps, we see that any homotopy moment map restricts to a weak moment map. However, note that there are n equations in the definition of a weak moment map, while a homotopy moment map has to obey $n + 1$ equations. Namely, it's the equation (3.16) that the weak moment map is not required to satisfy.

Note that when $n = 1$, Definition 4.1.1 gives the weak comoment map from Definition 2.4.8.

In [30] J. Herman uses the weak moment map to extend the results of [55] and draw further parallels with symplectic geometry.

Before proceeding further, we establish that in this chapter we assume the following set-up:

(M, ω) is an n-plectic manifold, Lie group G (or Lie algebra $\mathfrak{g}$) acts on (M, ω), and this action preserves ω.

We also recall the following result from algebraic topology that will be of great importance in what follows. We refer the reader to [24] for a detailed treatment.

Theorem 4.1.3 (H. Künneth)**.** *Let X and Y be cochain complexes of vector spaces. Then for every $n \in \mathbb{N}$, there is an isomorphism*

$$\varkappa : H^n(X \otimes Y) \to \bigoplus_k H^k(X) \otimes H^{(n-k)}(Y).$$

Remark 4.1.4. Note that there is an explicit expression for the inverse map $\varkappa^{-1}$ given by $\varkappa^{-1}([\alpha]\otimes[\beta]) = ([\alpha\otimes\beta])$, for $[\alpha]\in H^k(X)$ and $[\beta]\in H^{(n-k)}(Y)$.

4.2 Existence in terms of a double complex

In this subsection we formulate an existence criterion for weak moment maps that is analogous to Proposition 3.4.18.

We first need the following lemma which also explains why the Lie kernel is considered as the domain of weak moment maps: there is no way for the form $\iota_{v_x}\omega$ to be exact if it is not closed, and the closedness is guaranteed only for v_p such that p is in the Lie kernel, as demonstrated by the following Lemma.

Lemma 4.2.1. *([41, Lemma 2.4]) Let G act on an n-plectic manifold (M,ω), and let this action preserve ω. For $p\in P_{k,\mathfrak{g}}, k=1,...,n+1$, the form $\iota_{v_p}\omega$ is closed, i.e.,*

$$d\iota_{v_p}\omega = 0.$$

Proof. For $k=1$, we have $P_{1,\mathfrak{g}}=\mathfrak{g}$, and we need to show that $d\iota_{v_x}\omega=0$ for all $x\in\mathfrak{g}$. This follows from Cartan's magic formula:

$$\pounds_{v_x}\omega = d\iota_{v_x}\omega + \iota_{v_x}d\omega = d\iota_{v_x}\omega,$$

since ω is closed, and $\pounds_{v_x}\omega=0$.

For $k\geq 2$, consider $p=\sum x_{i_1}\wedge x_{i_2}\wedge\cdots\wedge x_{i_k}\in P_{k,\mathfrak{g}}$. By Lemma 3.3.26,

$d\iota(v_{x_{i_1}}\wedge ...\wedge v_{x_{i_k}})\omega =$

$$(-1)^k\sum_{1\leq l<m\leq k}(-1)^{i+j}\iota([v_{x_{i_l}},v_{x_{i_m}}]\wedge v_{x_{i_1}}\wedge ...\wedge \hat{v}_{x_{i_l}}\wedge ...\wedge\hat{v}_{x_{i_m}}\wedge ...\wedge v_{x_{i_k}})\omega,$$

because ω is closed, and the action of G acts preserves ω. Then, by linearity, for $p=\sum x_{i_1}\wedge x_{i_2}\wedge\cdots\wedge x_{i_k}$ we have

$$\begin{aligned} d\iota_{v_p}\omega &= \sum d\iota(v_{x_{i_1}}\wedge ...\wedge v_{x_{i_k}})\omega \\ &= \sum(-1)^k\sum_{1\leq l<m\leq k}(-1)^{i+j}\iota([v_{x_{i_l}},v_{x_{i_m}}]\wedge v_{x_{i_1}}\wedge ...\wedge \hat{v}_{x_{i_l}}\wedge ...\wedge\hat{v}_{x_{i_m}}\wedge ...\wedge v_{x_{i_k}})\omega \\ &= (-1)^k\iota(v_{\delta_k p})\omega = 0, \end{aligned}$$

since $p \in P_{k,\mathfrak{g}}$ and therefore $\delta_k p = 0$.

□

Remark 4.2.2. Note that when $n = 1$ we obtain Lemma 2.3.10.

From the above lemma we conclude that $\iota_{v_p}\omega$ for $p \in P_{k,\mathfrak{g}}$ defines a class in the de Rham cohomology of M, so the maps $p \mapsto \iota_{v_p}\omega$ induce well-defined maps $P_{k,\mathfrak{g}} \to H^{n+1-k}(M)$, and we can state the following lemma:

Lemma 4.2.3. *A weak moment map exists if and only if the maps*

$$P_{k,\mathfrak{g}} \to H^{n+1-k}(M)$$

$$p \mapsto [\iota_{v_p}\omega]$$

are identically zero for $1 \leq k \leq n$.

Proof. By definition of weak moment maps, for all $p \in P_{k,\mathfrak{g}}$ and all k, the form $\iota_{v_p}\omega$ has to be exact. □

We can now state the result analogous to that of Proposition 3.4.18. Define $P_{\geq 1,\mathfrak{g}} := \bigoplus_{k=1}^{dim\,\mathfrak{g}} P_{k,\mathfrak{g}}$, and consider the double complex

$$(P^*_{\geq 1,\mathfrak{g}} \otimes \Omega(M), 0, d) \tag{4.1}$$

with the zero differential on $P^*_{\geq 1,\mathfrak{g}}$ and the de Rham differential on M.

Let $\widehat{C}$ be the total complex of this double complex with the differential

$$\widehat{d}_{tot} := 1 \otimes d \tag{4.2}$$

Define $\widehat{\omega} \in \widehat{C}$ as

$$\widehat{\omega} := \sum_{k=1}^{n} (-1)^{k-1} (\omega_k|_{P_{k,\mathfrak{g}}}), \tag{4.3}$$

where the ω_k are as defined in (3.33), i.e.,

$$\omega_k : \wedge^k \mathfrak{g} \to \Omega^{n+1-k}$$

$$(x_1, ..., x_k) \mapsto \iota(v_{x_1} \wedge ... \wedge v_{x_k})\omega,$$

where v_{x_i} is the fundamental vector field corresponding to $x_i \in \mathfrak{g}$.

We then have the following:

Proposition 4.2.4. *Let* $\widehat{\alpha} := \widehat{\alpha}_1 + ... + \widehat{\alpha}_n$*, with* $\widehat{\alpha}_k \in P^*_{k,\mathfrak{g}} \otimes \Omega^{n-k}(M)$*. Then* $\widehat{d}_{tot}\widehat{\alpha} = \widehat{\omega}$ *if and only if*

$$\widehat{f_k} := \zeta(k)\widehat{\alpha}_k : P_{k,\mathfrak{g}} \to \Omega^{n-k}(M), \qquad k = 1, ..., n$$

are the components of a weak moment map for the action of G *on* (M, ω)*. I.e., the existence of a weak moment map is equivalent to the vanishing of* $[\widehat{\omega}] \in H^{n+1}(\widehat{C})$*.*

Proof. The proof follows from Definition 4.1.1. □

4.3 Relation to homotopy moment map

By Remark 4.1.2, every homotopy moment map induces a weak moment map by restriction to the Lie kernel, i.e., if there is a homotopy moment map for a given Lie algebra action, there is also a weak moment map. A natural question to ask is, what about the converse implication? I.e., if a given group action on an n-plectic manifold admits a weak moment map, does it also admit a homotopy moment map?

The example of symplectic geometry already shows that the answer to the above question is negative. Indeed, in symplectic geometry, a weak moment map for a given action exists if the action is by Hamiltonian vector fields. However, as we saw in Example 2.4.22 and Example 2.4.24, such an action does not always admit a homotopy moment map.

Nonetheless, we will see in this section that adding a certain condition necessary for the existence of a homotopy moment map resolves the problem.

First we establish the following simple, but important result.

Lemma 4.3.1. *If there exists a homotopy moment map for* $\mathfrak{g}$ *acting on* (M, ω)*, then the map*

$$\phi : P_{n+1,\mathfrak{g}} \to C^\infty(M)$$

$$p \mapsto \iota_{v_p}\omega$$

vanishes identically.

Proof. If there exists a homotopy moment map f for the action of $\mathfrak{g}$ on (M, ω), then it has to satisfy equation (3.16)

$$-\widetilde{f_n}(\delta p) = \zeta(n+1)\iota_{v_p}\omega$$

for all $p \in \Lambda^{n+1}\mathfrak{g}$. This means that for $p \in P_{n+1,\mathfrak{g}}$, we have $\iota_{v_p}\omega = 0$. □

Recall the double complex $(\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M), d_{\mathfrak{g}}, d)$ introduced in see (3.31). In this chapter we will denote the total complex of this double complex by $\widetilde{C}$ and its total differential by $\widetilde{d}_{tot}$. Remember that, by Proposition 3.4.18, a homotopy moment map for the action of $\mathfrak{g}$ on (M, ω) exists if and only if the class $[\widetilde{\omega}] = 0$ in $H^{n+1}(\widetilde{C})$ for $\widetilde{\omega}$ defined in (3.34).

By the Künneth theorem, $H^{n+1}(\widetilde{C}) = \bigoplus_{i=1}^{n+1} H^i(\mathfrak{g}) \otimes H^{n+1-i}(M)$, where $H^i(\mathfrak{g})$ are the Lie algebra cohomology groups and $H^{n+1-i}(M)$ the de Rham cohomology groups. In particular, the class $[\widetilde{\omega}]$ can be decomposed into components in $H^i(\mathfrak{g}) \otimes H^{n+1-i}(M)$. It turns out that ϕ defined in Lemma 4.3.1 describes the component of $[\widetilde{\omega}]$ in $H^{n+1}(\mathfrak{g}) \otimes H^0(M)$.

To see this, recall the following map from (3.28):

$$c_m^{\mathfrak{g}} : \Lambda^{n+1}\mathfrak{g} \to \mathbb{R}, \quad p \mapsto (-1)^n \zeta(n+1)\iota_{v_p}\omega|_m, \tag{4.4}$$

for some $m \in M$. Recall from Lemma 3.4.11 that this map is a $(n+1)$-cocycle in the Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in $\mathbb{R}$, and, by [7, Cor. 9.3], if M is connected, the cohomology class $[c_m^{\mathfrak{g}}] \in H^{n+1}(\mathfrak{g})$ does not depend on the point $m \in M$. Moreover, by Proposition 3.4.12, if a connected n-plectic manifold (M, ω) is equipped with a G-action which induces a homotopy moment map, then

$$[c_m^{\mathfrak{g}}] = 0.$$

Up to sign, the class $[c_m^{\mathfrak{g}}]$ can be interpreted as the evaluation at m of the $H^{n+1}(\mathfrak{g}) \otimes H^0(M)$ component of $[\widetilde{\omega}]$.

We have the following lemma:

Lemma 4.3.2. *Let (M, ω) be a connected n-plectic manifold equipped with a G-action preserving ω. Let $c_m^{\mathfrak{g}}$ be defined as in equation* (4.4)*, and ϕ be defined as in Lemma 4.3.1. Then the condition*

$$[c_m^{\mathfrak{g}}] = 0 \in H^{n+1}(\mathfrak{g})$$

is equivalent to $\phi \equiv 0$.

Proof. Indeed, assume $\phi \equiv 0$. Then, for $p \in P_{n+1,\mathfrak{g}}$

$$c_m^{\mathfrak{g}}(p) = (-1)^n \zeta(n+1)(\iota_{v_p}\omega)|_m = 0$$

for any $p \in M$. That means that $c_m^{\mathfrak{g}} \in (P_{n+1,\mathfrak{g}})^\circ$, where $(P_{n+1,\mathfrak{g}})^\circ$ is the annihilator of $P_{n+1,\mathfrak{g}}$. Therefore $c_m^{\mathfrak{g}} \in (P_{n+1,\mathfrak{g}})^\circ = (ker\, \delta_{n+1})^\circ = im\, d_{\mathfrak{g}}^n$, i.e., $[c_m^{\mathfrak{g}}] = 0$. Note that we have used that:

- $d^n_{\mathfrak{g}} : \wedge^n \mathfrak{g}^* \to \wedge^{n+1}\mathfrak{g}^*$ is dual to $\delta_{n+1} : \wedge^{n+1}\mathfrak{g} \to \wedge^n \mathfrak{g}$ and
- the linear algebra fact that the annihilator of the kernel of a linear map is the image of the dual map.

Conversely, suppose $[c^{\mathfrak{g}}_m] = 0$. Since M is connected, this holds at all $m \in M$. Then $c^{\mathfrak{g}}_m = d_{\mathfrak{g}}\xi$ for some $\xi \in \Lambda^n \mathfrak{g}^*$. For any $p \in P_{n+1,\mathfrak{g}}$ and $m \in M$ this means

$$\iota_{v_p}\omega|_m = (-1)^n \zeta(n+1)c^{\mathfrak{g}}_m(p) = (-1)^n\zeta(n+1)(d_{\mathfrak{g}}\xi)(p) = (-1)^n\zeta(n+1)\xi(\delta p) = 0.$$

□

We can now state the central result of this chapter:

Theorem 4.3.3. *Let (M,ω) be an n-plectic manifold, and let $\mathfrak{g}$ act on (M,ω) by preserving ω. The following statements are equivalent:*

1. *The action of $\mathfrak{g}$ on (M,ω) admits a homotopy moment map*
2. *The action of $\mathfrak{g}$ on (M,ω) admits a weak moment map and $\phi \in P^*_{n+1,\mathfrak{g}} \otimes C^\infty(M)$ as defined in Lemma 4.3.1 vanishes identically.*

Proof. The implication (1) → (2) is an immediate consequence of the fact that any homotopy moment map restricts to a weak moment map and Lemma 4.3.1. To prove the converse, we first observe that $\widetilde{\omega}|_{P_{\geq 1,\mathfrak{g}}} = \widehat{\omega} + \phi$. In other words, the restriction

$$(res \otimes id) : \widetilde{C} = \Lambda^{\geq 1}\mathfrak{g}^* \otimes \Omega(M) \to P^*_{\geq 1,\mathfrak{g}} \otimes \Omega(M) = \widehat{C} \tag{4.5}$$

maps $\widetilde{\omega}$ to $\widehat{\omega} + \phi$, where $\widehat{\omega}$ is defined by formula (4.3). We claim that this restriction is a chain map. Indeed, for $\xi \otimes \alpha \in \wedge^k \mathfrak{g}^* \otimes \Omega^l(M)$ we have

$$(res \otimes id)\xi \otimes \alpha = \xi|_{P_{\mathfrak{g}}} \otimes \alpha,$$

and therefore

$$\begin{aligned} &(res \otimes id)(d_{tot}(\xi \otimes \alpha)) = \\ &(res \otimes id)(d_{\mathfrak{g}}\xi \otimes \alpha + (-1)^k \xi \otimes d\alpha) = \\ &(d_{\mathfrak{g}}\xi)|_{P_{\mathfrak{g}}} \otimes \alpha + (-1)^k \xi|_{P_{\mathfrak{g}}} \otimes d\alpha = 0 + (-1)^k\xi|_{P_{\mathfrak{g}}} \otimes d\alpha \\ &= \widehat{d}_{tot}((res \otimes id)(\xi \otimes \alpha)) \end{aligned}$$

by definition of $P_{\mathfrak{g}}$, and by definition of $\widehat{d}_{tot}$ in (4.2).

Note that by Proposition 4.2.4 and the assumptions of item 2. in the statement of the Theorem, we have $[\widehat{\omega}+\phi]=[\widehat{\omega}]=0\in H^{n+1}(\widehat{C})$, where $\widehat{\omega}$ is defined by formula (4.3). Therefore, if we can prove that the induced map in cohomology $[res\otimes id]:H(\widetilde{C})\to H(\widehat{C})$ is injective, then the preimage $[\widetilde{\omega}]$ of $[\widehat{\omega}]=0$ will also vanish, and therefore, by 3.4.18, we can conclude that there exists a homotopy moment map.

To prove injectivity, we first note that the sequence

$$0\to P_{k,\mathfrak{g}}\xrightarrow{i}\wedge^k\mathfrak{g}\xrightarrow{\delta^k}\wedge^{k-1}\mathfrak{g}$$

and the dual sequence

$$0\leftarrow P^*_{k,\mathfrak{g}}\xleftarrow{\pi}\wedge^k\mathfrak{g}^*\xleftarrow{d^{k-1}_{\mathfrak{g}}}\wedge^{k-1}\mathfrak{g}^*$$

are exact.

Therefore,

$$P^*_{k,\mathfrak{g}}=\wedge^k\mathfrak{g}^*/im(d^{k-1}_{\mathfrak{g}})\hookleftarrow ker(d^k_{\mathfrak{g}})/im(d^{k-1}_{\mathfrak{g}})=H^k(\mathfrak{g}),$$

i.e., the following map is injective:

$$\begin{aligned}[res]\otimes[id]:H^k\mathfrak{g}\otimes H^{n+1-k}(M)&\to P^*_{k,\mathfrak{g}}\otimes H^{n+1-k}(M)\\ [\xi]\otimes[\alpha]&\mapsto \xi|_{P_{\mathfrak{g}}}\otimes[\alpha].\end{aligned}$$

Consider the following diagram:

$$\begin{array}{ccc}\bigoplus_{k\geq 1}H^k\mathfrak{g}\otimes H^{n+1-k}(M) & \xrightarrow{[res]\otimes[id]} & \bigoplus_{k\geq 1}P^*_{k,\mathfrak{g}}\otimes H^{n+1-k}(M)\\ \varkappa\uparrow & & \uparrow\varkappa\\ H^{n+1}(\wedge^{\geq 1}\mathfrak{g}^*\otimes\Omega(M)) & \xrightarrow[{[res\otimes id]}]{} & H^{n+1}(P^*_{\geq 1,\mathfrak{g}}\otimes\Omega(M))\end{array}$$

Here the map $\varkappa$ is the Künneth isomorphism, and the bottom map $[res\otimes id]$ is the one induced in cohomology from the map $(res\otimes id)$ defined in (4.5). Note that, since $\varkappa$ is an isomorphism, and $[res] \otimes [id]$ is injective, the composition $([res] \otimes [id]) \circ \varkappa$ is also injective. Thus, if we can show that the diagram above commutes, it will follow that $[res \otimes id]$ is injective.

Note that the above diagram commuting is equivalent to the following diagram commuting:

$$\begin{array}{ccc} \bigoplus_{k\geq 1} H^k\mathfrak{g} \otimes H^{n+1-k}(M) & \xrightarrow{[res]\otimes[id]} & \bigoplus_{k\geq 1} P^*_{k,\mathfrak{g}} \otimes H^{n+1-k}(M) \\ \downarrow \varkappa^{-1} & & \downarrow \varkappa^{-1} \\ H^{n+1}(\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M)) & \xrightarrow[{[res\otimes id]}]{} & H^{n+1}(P^*_{\geq 1,\mathfrak{g}} \otimes \Omega(M)) \end{array}$$

In other words, we have to check if

$$\varkappa^{-1}(([res] \otimes [id])([\alpha] \otimes [\gamma])) = [res \otimes id](\varkappa^{-1}([\alpha] \otimes [\gamma])), \tag{4.6}$$

for $[\alpha] \otimes [\gamma] \in H^k\mathfrak{g} \otimes H^{n+1-k}(M)$.

On the left-hand side of equation (4.6), we have

$$\begin{aligned} \varkappa^{-1}(([res] \otimes [id])([\alpha] \otimes [\gamma])) &= \varkappa^{-1}(\alpha|_{P_{k,\mathfrak{g}}} \otimes [\gamma]) \\ &= [\alpha|_{P_{k,\mathfrak{g}}} \otimes \gamma] \end{aligned}$$

On the right-hand side of equation (4.6), we have:

$$\begin{aligned} [res \otimes id](\varkappa^{-1}([\alpha] \otimes [\gamma])) &= [res \otimes id][\alpha \otimes \gamma] \\ &= [\alpha|_{P_{k,\mathfrak{g}}} \otimes \gamma], \end{aligned}$$

so both sides of equation (4.6) agree, and therefore diagrams above commute, and the map $[res \otimes id]$ is injective.

□

Proposition 4.2.4 characterized existence of weak moment maps in terms of the complex $\widehat{C}$. The following corollary characterizes existence of weak moment maps in terms of the complex $\widetilde{C}$ that is responsible for existence of homotopy moment maps.

Corollary 4.3.4. *A weak moment exists if and only if the projection of* $[\widetilde{\omega}]$ *to*

$$\bigoplus_{k=1}^{n} H^k(\mathfrak{g}) \otimes H^{n-k+1}(M)$$

vanishes.

Proof. Let p_i^k, where $i = 1, ..., dim\ P_{k,\mathfrak{g}}$, be a basis of $P_{k,\mathfrak{g}}$ for $k = 1, ..., n$. Then $\omega_k|_{P_{k,\mathfrak{g}}}$ can be written as $\omega_k|_{P_{k,\mathfrak{g}}} = \sum\limits_i (p_i^k)^* \otimes \omega_k(p_i^k)$, where $(p_i^k)^*$ are members of the dual basis of $P_{k,\mathfrak{g}}^*$. Note that, since each $\omega_k(p_i^k)$ is closed by Lemma 4.2.1, each $\omega_k|_{P_{k,\mathfrak{g}}}$ is also closed under $\widehat{d}_{tot} = 1 \otimes d$. Thus, the image of $[\widehat{\omega} + \phi] \in H^{n+1}(\widehat{C})$ under the Künneth isomorphism $\varkappa$ is

$$\varkappa([\widehat{\omega} + \phi]) = \sum_{k=1}^{n} (-1)^{k-1} \varkappa[\omega_k|_{P_{k,\mathfrak{g}}}] + (-1)^n \varkappa[\phi]$$

so $\varkappa([\widehat{\omega} + \phi])$ can be divided into two parts:

$$\varkappa([\widehat{\omega}]) = \sum_{k=1}^{n} (-1)^{k-1} \varkappa[\omega_k|_{P_{k,\mathfrak{g}}}] \in \bigoplus_{k=1}^{n} P_{k,\mathfrak{g}^*} \otimes H^{n-k+1}(M)$$

and

$$(-1)^n \varkappa[\phi] \in P_{n+1,\mathfrak{g}^*} \otimes H^0(M).$$

By the commutativity of the diagram in the proof of Theorem 4.3.3, we have

$$[res] \otimes [id](\varkappa[\widetilde{\omega}]) = \varkappa([\widehat{\omega}]) + (-1)^n \varkappa([\phi]).$$

It is then clear from the nature of the map $[res] \otimes [id]$ that the preimage of $\varkappa([\widehat{\omega}])$ under $[res] \otimes [id]$ is the projection of $\varkappa([\widetilde{\omega}])$ to $\bigoplus_{k=1}^{n} H^k(\mathfrak{g}) \otimes H^{n-k+1}(M)$. Since we showed in the proof of Theorem 4.3.3 that the map $[res] \otimes [id]$ is injective, $\varkappa([\widehat{\omega}])$ vanishes if and only if that preimage vanishes. Noting that the Künneth map is an isomorphism, we get the statement of the corollary.

□

With this corollary, we can recover the following result from [29]:

Proposition 4.3.5. *[29, Prop. 5.12] If* $H^1(\mathfrak{g}) = ... = H^n(\mathfrak{g}) = 0$*, then a weak moment map exists.*

We turn to examples to illustrate Theorem 4.3.3. The first example deals with the more familiar symplectic case:

Example 4.3.6. Let $n = 1$, i.e., consider a connected Lie group G acting on a connected symplectic manifold (M, ω). In this case, a weak moment map is a map $\widehat{f} : \mathfrak{g} \to C^\infty(M)$ that satisfies

$$d(\widehat{f}(x)) = -\iota_{v_x}\omega,$$

and a homotopy moment map is a map $\widetilde{f} : \mathfrak{g} \to C^\infty(M)$ that satisfies

$$d(\widetilde{f}(x)) = -\iota_{v_x}\omega$$
$$\widetilde{f}([x, y]) = \{\widetilde{f}(x), \widetilde{f}(y)\} = \omega(v_x, v_y).$$

In this thesis, we call the latter map simply a *moment map*. As was stated in Proposition 2.4.19, if there exists a weak moment map, then the obstruction to the existence of an equivariant moment map lies in $H^2(\mathfrak{g})$.

More specifically, let $\widehat{f}$ be a weak moment map[2]. Consider

$$\begin{aligned} h(x, y) :&= \{\widehat{f}(x), \widehat{f}(y)\} - \widehat{f}([x, y]) \\ &= \omega(v_x, v_y) - \widehat{f}([x, y]). \end{aligned}$$

Since $d(\omega(v_x, v_y)) = -\iota_{[v_x,v_y]}\omega = d(\widehat{f}([x, y]))$, it follows that $h(x, y)$ is a constant function on M, and therefore it defines an element $h \in \wedge^2\mathfrak{g}^*$. Evaluating $h(x, y)$ at any point $m \in M$, we get

$$\begin{aligned} h(x, y) &= \omega(v_x, v_y)|_m - \widehat{f}|_m([x, y]) \\ &= c^{\mathfrak{g}}_m(x, y) + d_{\mathfrak{g}}\widehat{f}(x, y)|_m. \end{aligned}$$

[2] What follows is basically the content of the second half of the proof of Theorem 3.4.13, adjusted to $n = 1$.

If we assume that $c_m^{\mathfrak{g}}$ is exact, then $h \in \wedge^2\mathfrak{g}^*$ is exact, i.e., there exists $b \in \mathfrak{g}^*$ such that $h = d_{\mathfrak{g}}b$. Then $\widetilde{f} := \widehat{f} - b$ is an equivariant moment map. Indeed,

$$\begin{aligned}\widetilde{f}([x,y]) - \omega(v_x, v_y) &= \widehat{f}([x,y]) - b([x,y]) - \omega(v_x, v_y) \\ &= \widehat{f}([x,y]) + d_{\mathfrak{g}}b(x,y) - \omega(v_x, v_y) \\ &= \widehat{f}([x,y]) + h(x,y) - \omega(v_x, v_y) \\ &= 0\end{aligned}$$

Note that, by Lemma 4.3.2, the exactness of $c_m^{\mathfrak{g}}$ is equivalent to the vanishing of ϕ. Thus, if there exists a weak moment map for a Lie group acting on a symplectic manifold, then there exists a homotopy moment map if and only if $\phi \equiv 0$, i.e., we recover Theorem 4.3.3 in the special case of $n = 1$.

Thus, the example above yields many instances in symplectic geometry where a weak moment map exists, but a homotopy moment map does not (again, see Example 2.4.22 and Example 2.4.24). The next examples illustrate such cases in n-plectic geometry.

Example 4.3.7. Let G be a connected compact simple group acting on itself by multiplication, and $\omega = \langle \theta^L, [\theta^L, \theta^L] \rangle$ as in Example 3.3.8 and Example 3.4.16. We saw in Example 3.4.16 that ω is invariant under left translations of G, but the class $[c_p^{\mathfrak{g}}]$ does not vanish, so there is no homotopy moment map for this action. However, since $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$ (see, e.g., [32]), a weak homotopy moment map exists for this action, by Proposition 4.3.5.

Example 4.3.8. Let V be a real finite-dimensional vector space acting on itself by translations, and let ω be a closed invariant differential form on V obtained by translating a nonzero element of $\wedge^{n+1}V^*$. We saw in Example 3.4.15 that the cocycle $c_p^{\mathfrak{g}}$ defined by ω doesn't vanish, and therefore there is no homotopy moment map for this action. However, since V is a vector space, all $H^k(M)$ vanish for $k \geq 1$, and therefore, by Lemma 4.2.3, this action admits a weak moment map.

In the next example we will explicitly construct a 2-plectic homotopy moment map from a weak moment map.

Example 4.3.9. Consider the action of $SO(3)$ on $(M = \mathbb{R}^3, \omega = dx_1 \wedge dx_2 \wedge dx_3)$ by rotations. Note that, since $H^1(M) = 0$, the first component of a weak moment map for this action exists. The Lie algebra $\mathfrak{so}(3)$ is spanned by the following elements:

$$e_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

These elements satisfy the following bracket relations:

$$[e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_2, e_3] = e_1,$$

therefore, $P_{2,\mathfrak{so}(3)} \equiv 0$, , and a weak homotopy moment map for this action has only one component. It also follows from the above bracket relations that $P_{3,\mathfrak{so}(3)} = \wedge^3 \mathfrak{so}(3)$, which we will need later. Note further, that the orbits of this action are of dimensions 0 and 2, hence $\phi \in P^*_{3,\mathfrak{so}(3)} \otimes C^\infty(M) = \wedge^3 \mathfrak{so}(3)^* \otimes C^\infty(M)$ defined in Lemma 4.3.1 vanishes. Therefore, by Theorem 4.3.3, there exists a homotopy moment map for this action.

The fundamental vector fields v_i corresponding to the elements e_i are given by

$$v_1 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}, \; v_2 = x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}, \; v_3 = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}.$$

Defining

$$\widehat{f}_1(e_1) = \omega(v_2, v_3), \; \widehat{f}_1(e_2) = -\omega(v_1, v_3), \; \widehat{f}_1(e_3) = \omega(v_1, v_2)$$

on basis elements gives the weak moment map.

Note that $\widetilde{f} = (\widetilde{f}_1 = \widehat{f}_1, \widetilde{f}_2 \equiv 0)$ is a preimage of $\widehat{f}$ under the map (4.5). By Proposition 4.2.4, $\widehat{d}_{tot}(\widehat{f}) = \widehat{\omega}$. The map (4.5) is a chain map, and thus it maps $\widetilde{d}_{tot}(\widetilde{f})$ to $\widehat{d}_{tot}(\widehat{f}) = \widehat{\omega}$. Since $\phi = 0$, the image of $\widetilde{\omega}$ under the map (4.5) is $\widehat{\omega}$, i.e., $\widetilde{d}_{tot}(\widetilde{f})$ and $\widetilde{\omega}$ have the same image under the map (4.5). Since $\widetilde{\omega}$ and $\widetilde{d}_{tot}(\widetilde{f})$ are both $\widetilde{d}_{tot}$-closed, so is their difference $\widetilde{\psi} := \widetilde{\omega} - \widetilde{d}_{tot}\widetilde{f}$. The induced map in cohomology maps the class of $\widetilde{\psi}$ in $H^{n+1}(\widetilde{C})$ to the 0 class in $H^{n+1}(\widehat{C})$, and since we showed that this map is injective, this means that the class of $\widetilde{\psi}$ is 0, i.e., $\widetilde{\psi}$ is $\widetilde{d}_{tot}$-exact. The sum of any primitive of $\widetilde{\psi}$ with $\widetilde{f}$ gives a primitive of $\widetilde{\omega}$, i.e., a homotopy moment map. So, to construct a homotopy moment map, we need to find a primitive of $\widetilde{\psi}$, i.e., find a $\widetilde{h} = (h_1, h_2) \in \widetilde{C}^2$ that satisfies the following equations:

$$-dh_1 = 0$$

$$d_{\mathfrak{g}} h_1 + dh_2 = -\omega_2 - d_{\mathfrak{g}} \widehat{f}_1$$

$$d_{\mathfrak{g}} h_2 = \omega_3.$$

Note that $\omega_3 = 0$. Also, evaluating $-\omega_2 - d_{\mathfrak{g}}\widehat{f_1}$ on basis elements, we get, using the definition of $\widehat{f_1}$ and bracket relations between the e_i,

$$\omega_2(e_i, e_j) - f_1([e_i, e_j]) = 0.$$

Therefore the equations above become:

$$-dh_1 = 0$$

$$d_{\mathfrak{g}}h_1 + dh_2 = 0$$

$$d_{\mathfrak{g}}h_2 = 0.$$

Note that the last equation is satisfied for any $h_2 \in \wedge^2\mathfrak{so}(3)^* \otimes C^\infty(M)$, since for all $x, y, z \in \mathfrak{g}$, we have $(d_{\mathfrak{g}}h_2)(x, y, z) = h_2(\delta_3(x, y, z))$, and $P_{3,\mathfrak{so}(3)} = \wedge^3\mathfrak{so}(3)$. Therefore, the equations above become:

$$-dh_1 = 0$$

$$d_{\mathfrak{g}}h_1 + dh_2 = 0.$$

It is easy to see that a trivial map $h_1 \equiv 0, h_2 \equiv 0$ satisfies this equation, and therefore, $\widetilde{f} = (\widetilde{f_1} = \widehat{f_1}, \widetilde{f_2} \equiv 0)$ is a homotopy moment map for this action.

Remark 4.3.10. Note that since in this example $H^i(M) = 0$ for $1 \leq i \leq n-1$, and $[c_m^{\mathfrak{g}}] = 0$, the existence of a homotopy moment map is also guaranteed by Theorem 3.4.13. Moreover, it was stated in Example 3.4.5 that a homotopy moment map can be constructed for the action of $SO(n)$ on $\mathbb{R}^n$ using equations (3.18). However, the homotopy moment map constructed in the above example differs from the one constructed from equations (3.18).

4.4 Strict extensions

By Theorem 4.3.3, assuming $\phi \equiv 0$, the existence of weak moment maps implies the existence of homotopy moment maps. This raises the following question: Given a weak moment map and assuming $\phi \equiv 0$, does there always exist a homotopy moment map that restricts to the given weak moment map? The following proposition answers this question:

Proposition 4.4.1. *Let $\widehat{f}$ be a weak moment map, and $\phi = 0$. There exists a well-defined class $[\gamma]_{\widetilde{d}_{tot}} \in H^{n+1}(\widetilde{C})$ such that the following are equivalent:*

1. $[\gamma]_{\widetilde{d}_{tot}} = 0$ *and γ admits a $\widetilde{d}_{tot}$-primitive in* $\bigoplus_{k=1}^n d_{\mathfrak{g}}(\Lambda^k\mathfrak{g}^*) \otimes \Omega^{n-k-1}(M)$

2. *There exists a homotopy moment map* $\widetilde{f}$, *such that* $\widetilde{f}|_{P_{\mathfrak{g}}} = \widehat{f}$.

Proof. Let $\widehat{\alpha} = \widehat{\alpha}_1 + ... + \widehat{\alpha}_n \in \widehat{C}$ be a potential of $\widehat{\omega} \in \widehat{C}$ corresponding to $\widehat{f}$ under the bijection in Proposition 4.2.4. Let $\beta \in \widetilde{C}$ be any preimage of $\widehat{\alpha}$ under the map (4.5). Such a preimage exists as the restriction $\widetilde{C} \to \widehat{C}$ is surjective. However, it might not be a potential of $\widetilde{\omega}$. We can thus consider the element $\gamma = \widetilde{\omega} - \widetilde{d}_{tot}\beta \in \widetilde{C}$. Note that $\gamma \in ker(res \otimes id) = d_{\mathfrak{g}}(\wedge^{\geq 1}\mathfrak{g}^*) \otimes \Omega(M)$.

First of all, we will show that the class $[\gamma]_{\widetilde{d}_{tot}}$ does not depend on the choice of β. Indeed, let β and β' be 2 different preimages of $\widehat{a}$ under the map (4.5). Then $\beta' = \beta + b$ for some $b \in \ker(res \otimes id)$, and

$$\gamma' = \widetilde{\omega} - \widetilde{d}_{tot}\beta' = \widetilde{\omega} - \widetilde{d}_{tot}(\beta + b) = \widetilde{\omega} - \widetilde{d}_{tot}\beta - \widetilde{d}_{tot}b = \gamma - \widetilde{d}_{tot}b,$$

i.e., $[\gamma]_{\widetilde{d}_{tot}} = [\gamma']_{\widetilde{d}_{tot}}$. Moreover, if $[\gamma]_{\widetilde{d}_{tot}} = 0$, then γ admitting a primitive in $d_{\mathfrak{g}}(\wedge^i\mathfrak{g}) \otimes \Omega^{n-i}(M)$ does not depend on the choice of β. Indeed, if $\gamma = \widetilde{\omega} - \widetilde{d}_{tot}\beta = \widetilde{d}_{tot}\mu$ for $\mu \in d_{\mathfrak{g}}(\wedge^{\geq 1}\mathfrak{g}^*) \otimes \Omega(M)$, then choosing a $\beta' = \beta + b$ yields

$$\gamma' = \widetilde{\omega} - \widetilde{d}_{tot}(\beta') = \widetilde{\omega} - \widetilde{d}_{tot}(\beta + b) = \widetilde{d}_{tot}\mu - \widetilde{d}_{tot}b = \widetilde{d}_{tot}(\mu - b).$$

Now note that $b \in \ker(res \otimes id) = d_{\mathfrak{g}}(\wedge^{\geq 1}\mathfrak{g}^*) \otimes \Omega(M)$.

To show that (2) $\Rightarrow$ (1), let's assume there exists a β that corresponds to a homotopy moment map, restricting to $\widehat{\alpha}$. Then, by Proposition 3.4.18, $\gamma = \widetilde{\omega} - \widetilde{d}_{tot}\beta = 0$, i.e., $\gamma = 0$ is the γ we need.

Conversely, assume that for some β, $\gamma = \widetilde{\omega} - \widetilde{d}_{tot}\beta = \widetilde{d}_{tot}\mu$ for $\mu \in d_{\mathfrak{g}}(\wedge^{\geq 1}\mathfrak{g}^*) \otimes \Omega(M)$. Then, $\widetilde{\omega} = \widetilde{d}_{tot}(\beta + \mu)$, i.e., $\beta + \mu$ corresponds to a homotopy moment map that restricts to $\widehat{f}$, since $\mu \in ker(res \otimes id)$.

□

Remark 4.4.2. Denote by γ_{i+1} the component of γ in $d_{\mathfrak{g}} \wedge^i \mathfrak{g}^* \otimes \Omega^{n-i}(M)$. Note that, since $\widetilde{d}_{tot} = 1 \otimes d$ on $d_{\mathfrak{g}}(\wedge^{\geq 1}\mathfrak{g}^*) \otimes \Omega(M)$, it follows from $d_{tot}\gamma = 0$ that $d\gamma_i = 0$ for all γ_i. Requiring γ to have a primitive $\mu \in d_{\mathfrak{g}}(\wedge^{\geq 1}\mathfrak{g}^*) \otimes \Omega(M)$ is equivalent to saying that each $\gamma_{i+1} = d\eta_i$, where $\eta_i \in d_{\mathfrak{g}} \wedge^i \mathfrak{g}^* \otimes \Omega^{n-i-1}(M)$.

Indeed, suppose $\gamma = \widetilde{d}_{tot}\mu$, where $\mu \in d_{\mathfrak{g}}(\wedge^{\geq 1}\mathfrak{g}^*) \otimes \Omega(M)$. Denote the component of μ in $d_{\mathfrak{g}} \wedge^i \mathfrak{g}^* \otimes \Omega^{n-i-1}(M)$ by μ_i. Then $\gamma_{i+1} = (-1)^{i+1}d\mu_i$.

Conversely, let each γ_{i+1} satisfy $\gamma_{i+1} = d\eta_i$ for some $\eta_i \in d_{\mathfrak{g}} \wedge^i \mathfrak{g}^* \otimes \Omega^{n-i-1}(M)$. Then $\widetilde{d}_{tot}(\sum_i (-1)^{i+1}\eta_i) = \gamma$.

Also note that $\gamma_{n+1} \in d_{\mathfrak{g}} \wedge^n \mathfrak{g}^* \otimes C^{\infty}(M)$, and therefore $\gamma_{n+1} = d\eta$ if and only if $\gamma_{n+1} = 0$.

Corollary 4.4.3. *Let G act on an n-plectic manifold M, let $\widehat{f}$ be a weak moment map for this action, and let γ be defined as in Proposition 4.4.1. If $H^i(M) = 0$ for $i \in \{1, ..., n-1\}$, then there exists a homotopy moment map restricting to $\widehat{f}$ if and only if $\gamma_{n+1} = 0$.*

Definition 4.4.4. Let $\widehat{f}$ be a weak moment map for an action of G on (M, ω). A homotopy moment map $\widetilde{f}$ is called a *strict extension* of $\widehat{f}$ if $\widetilde{f}|_{P_\mathfrak{g}} = \widehat{f}$, i.e., if $\widetilde{f}$ restricts to $\widehat{f}$.

Example 4.4.5. For $n = 1$, i.e., in symplectic geometry, $P_{1,\mathfrak{g}} = \mathfrak{g}$. Therefore, if a given weak moment map is not already a homotopy moment map, there is no homotopy moment map restricting to it.

To see this in terms of the results of Proposition 4.4.1, let $\widehat{f}$ be a symplectic weak moment map. Note that in this case $\gamma = \gamma_2$, and so by the Remark 4.4.2 and Proposition 4.4.1, there exists a homotopy moment map restricting to $\widehat{f}$ if and only

$$\begin{aligned}\gamma(x, y) &= -\omega_2(x, y) - d_\mathfrak{g}\widehat{f}(x, y)\\ &= -\omega(v_x, v_y) + \widehat{f}([x, y])\end{aligned}$$

vanishes, i.e., if and only if $\widehat{f}$ is already an equivariant moment map, i.e., a homotopy moment map.

For an example of a symplectic weak moment map that cannot be strictly extended to a homotopy moment map, consider a Lie algebra $\mathfrak{g}$ such that $H^1(\mathfrak{g}) = 0$. If there exists a homotopy moment map $\widetilde{f}$ for the action of $\mathfrak{g}$, then it is unique (see, e.g., [8, §26]). On the other hand, for an arbitrary nonzero $\xi \in \mathfrak{g}^*$ the map $\widehat{f} := \widetilde{f} + \xi$, satisfies the condition $d(\widehat{f}(x)) = -\iota_{v_x}\omega$ for all $x \in \mathfrak{g}$, i.e., is a weak moment map, but not a homotopy moment map.

Example 4.4.6. Consider the homotopy moment map $\widetilde{f}$ constructed in Example 4.3.9 for the action of $SO(3)$ on $(\mathbb{R}^3, \omega = dx_1 \wedge dx_2 \wedge dx_3)$, given by $\widetilde{f}_1(e_1) = \omega(v_2, v_3)$, $\widetilde{f}_1(e_2) = -\omega(v_1, v_3)$, $\widetilde{f}_1(e_3) = \omega(v_1, v_2)$ and $\widetilde{f}_2 \equiv 0$. This homotopy moment map coincided with the original weak moment map, i.e., in this case the original weak moment map admitted an obvious "strict extension" to a homotopy map. To see this in the context of Proposition 4.4.1, note that in this case

$$\gamma = \omega_1 - \omega_2 + \omega_3 - d_{\mathfrak{g}}\widetilde{f}_1 + d\widetilde{f}_1$$
$$= \omega_3$$
$$= 0,$$

since in this case $\omega_2(e_i, e_j) - \widetilde{f}_1([e_i, e_j]) = 0$ and $\omega_3 = \phi = 0$ (see the discussion in Example 4.3.9), i.e., $\gamma = \gamma_3$ vanishes in this example, and, by Proposition 4.4.1, there indeed exists a strict extension of the weak moment map to a homotopy moment map.

Moreover, any weak moment map for this action can be strictly extended to a homotopy moment map. Indeed, let $\widehat{f} = (\widehat{f}_1, \widehat{f}_2 \equiv 0)$ be a weak moment map for this action. The equations for a homotopy moment map are

$$d(\widetilde{f}_1(x)) = -\iota_{v_x}\omega$$
$$\widetilde{f}_1([x,y]) = d(\widetilde{f}_2(x,y)) + \omega(v_x, v_y)$$
$$-\widetilde{f}_2(\delta(x,y,z)) = -\omega(v_x, v_y, v_z).$$

Note that any $\widetilde{f}_2 \in \wedge^2\mathfrak{so}(3)^* \otimes C^\infty(M)$ restricts to $\widehat{f}_2 \equiv 0$, since $P_{2,\mathfrak{so}(3)} = 0$. Also, any $\widetilde{f}_2 \in \wedge^2\mathfrak{so}(3)^* \otimes C^\infty(M)$ satisfies the third equation above, since $P_{3,\mathfrak{so}(3)} = \wedge^3\mathfrak{so}(3)$, and $\omega_3 \equiv 0$, i.e., both sides of the third equation above are zero. Also note that, if the first one of the above equations is satisfied, then $d(\widetilde{f}_1([x,y])) = -\iota_{[v_x,v_y]}\omega = d(\omega(v_x, v_y))$, where the last equality is due to Lemma 3.3.26. Therefore the difference $\widetilde{f}_1([x,y]) - \omega(v_x, v_y)$ is a closed 1-form on $\mathbb{R}^3$ for all $x, y \in \mathfrak{so}(3)$. Since $H^1(\mathbb{R}^3) = 0$, this form is exact, and there exists a $\widetilde{f}_2$ satisfying the second of the equations above. Therefore, $\widetilde{f}_1 = (\widehat{f}_1, \widetilde{f}_2)$, is a homotopy moment map that restricts to the given weak moment map $\widehat{f} = (\widehat{f}_1, \widehat{f}_2 \equiv 0)$. This result is consistent with the Corollary 4.4.3 and the fact that $\gamma_3 = \omega_3 - d_{\mathfrak{g}}\widetilde{f}_2 = 0$, since both ω_3 and $d_{\mathfrak{g}}\widetilde{f}_2$ vanish.

4.5 Equivariance

Definition 4.5.1 ([7], [29])**.** Let G be a Lie group acting on an n-plectic manifold (M, ω), and let this action preserve ω. A homotopy moment map $f : \mathfrak{g} \to L_\infty(M, \omega)$ is called *equivariant* if for all $g \in G, p \in \wedge^k\mathfrak{g}$, and $1 \leq k \leq n$

$$f_k(Ad_g p) = \Phi^*_{g^{-1}} f_k(p), \tag{4.7}$$

where $\Phi : G \to \text{Diff(M)}$ denotes the action of G on (M,ω), and $\Phi_g^* : \Omega(M) \to \Omega(M)$ denotes the pullback of differential forms by Φ_g for all $g \in G$.

It is *infinitesimally equivariant* or $\mathfrak{g}$-*equivariant* if and only if for all $x \in \mathfrak{g}, p \in \wedge^k \mathfrak{g}$ and $1 \leq k \leq n$

$$f_k(ad_x p) - \pounds_{v_x} f_k(p) = 0, \tag{4.8}$$

where ad denotes the adjoint action of $\mathfrak{g}$ on $\wedge^k \mathfrak{g}$. In complete analogy, a weak homotopy moment map is *equivariant* if (4.7) holds for all $p \in P_{k,\mathfrak{g}}$ resp. *infinitesimally equivariant* if (4.8) holds for all $x \in \mathfrak{g}, p \in P_{k,\mathfrak{g}}$ and $1 \leq k \leq n$.

Remark 4.5.2. Similarly to the proof of Proposition 2.4.7, it can be shown that for a connected Lie group G, a homotopy (or weak) moment map is equivariant if and only if it is infinitesimally equivariant. We will treat the case of infinitesimal equivariance in the sequel, the equivariant working in complete analogy.

Consider the complex $\widetilde{C}^{\mathfrak{g}} = (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M))^{\mathfrak{g}}$, consisting of all $\mathfrak{g}$-invariant elements of $\widetilde{C}$ (recall that in this chapter we denote by $\widetilde{C}$ the total complex of (3.31)). The total differential $\widetilde{d}_{tot}$ restricts to $\widetilde{C}^{\mathfrak{g}}$, because $d_{\mathfrak{g}}$ commutes with the coadjoint action, and d commutes with the Lie derivative. Since the adjoint action $ad : \mathfrak{g} \to End(\wedge \mathfrak{g})$ preserves the subspace of δ-closed elements, it defines an action on $P_{\mathfrak{g}}$ and thus on $\widehat{C} = P_{\mathfrak{g}}^* \otimes \Omega(M)$. Again, the total differential $\widehat{d}_{tot}$ restricts to a differential on $\widehat{C}^{\mathfrak{g}} = (P_{\mathfrak{g}}^* \otimes \Omega(M))^{\mathfrak{g}}$, i.e., the set of all $\mathfrak{g}$-invariant elements of $\widehat{C}$. The coadjoint action of $\mathfrak{g}$ descends to $P_{\mathfrak{g},k}^*$, since $P_{\mathfrak{g},k}^* = \frac{\wedge^k \mathfrak{g}^*}{d_{\mathfrak{g}} \Lambda^{k-1} \mathfrak{g}^*}$, and $d_{\mathfrak{g}}(ad_x^* \xi) = ad_x^* d_{\mathfrak{g}} \xi$, hence we can define $ad_x^*[\xi] := [ad_x^* \xi]$.

Lemma 4.5.3. $\widetilde{f}_k \in \wedge^k \mathfrak{g}^* \otimes \Omega(M)$ *is infinitesimally equivariant iff* $f_k \in (\wedge^k \mathfrak{g}^* \otimes \Omega(M))^{\mathfrak{g}}$. *The same is true for* $\widehat{f}_k \in P_{\mathfrak{g},k}^* \otimes \Omega(M)$.

Proof. This follows straight from definitions: $\xi \otimes \alpha \in (\Lambda^k \mathfrak{g}^* \otimes \Omega(M))^{\mathfrak{g}}$ iff $x(\xi \otimes \alpha) = 0$ for all $x \in \mathfrak{g}$, i.e., for all $x \in \mathfrak{g}$, $ad_x^* \xi \otimes \alpha + \xi \otimes \pounds_{v_x} \alpha = 0$, which is equivalent to (4.8) when evaluated on $p \in \wedge^k \mathfrak{g}$. ☐

We state the following generalization of the well-known formula from Cartan calculus:

Lemma 4.5.4. *For any vector field* $v_i \in \mathfrak{X}(M)$ *and* $\tau \in \Omega(M)$ *any differential form, we have:*

$$\tau([v_1, v_2 \wedge ... \wedge v_l]) = \pounds_{v_1} \iota_{v_2 \wedge ... \wedge v_l} \tau - \iota_{v_2 \wedge ... \wedge v_l} \pounds_{v_1} \tau \tag{4.9}$$

Proof. When $l = 2$, we obtain the well-known formula $\iota_{[u,v]} = \mathcal{L}_u \iota_v - \iota_v \mathcal{L}_u$. The proof proceeds by induction on l. For details we refer the reader to [17, Prop. A.3] □

We can now show that $\widetilde{\omega}$ and $\widehat{\omega}$ are $\mathfrak{g}$-invariant elements of $\widetilde{C}$ and $\widehat{C}$, respectively:

Lemma 4.5.5. *The element* $\widetilde{\omega}$ *(resp.* $\widehat{\omega}$*) lies in* $\widetilde{C}^{\mathfrak{g}}$ *(resp.* $\widehat{C}^{\mathfrak{g}}$*).*

Proof. We will prove the statement for $\widetilde{\omega}$, the statement for $\widehat{\omega}$, follows as the image of a $\mathfrak{g}$-invariant element under the equivariant map $(res \otimes id)$ is necessarily $\mathfrak{g}$-invariant. It is enough to show that $\omega_k \in (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M))^{\mathfrak{g}}$. We need to show that

$$\omega_k(ad_x p) - \mathcal{L}_{v_x}\omega_k(p) = 0. \tag{4.10}$$

But this follows immediately from Lemma (4.5.4) and the assumption that the action of $\mathfrak{g}$ preserves ω. □

From the discussion above it follows that correspondences between potentials and moment maps established in Propositions 3.4.18 and 4.2.4 carry over to the $\mathfrak{g}$-equivariant setting and we have the following

Theorem 4.5.6. *Let* $\mathfrak{g}$ *act on* (M, ω) *by preserving* ω*. The action admits*

1. *a* $\mathfrak{g}$*-equivariant weak moment map if and only if* $[\widehat{\omega}] = 0 \in H^{n+1}(\widehat{C}^{\mathfrak{g}})$
2. *a* $\mathfrak{g}$*-equivariant homotopy moment map if and only if* $[\widetilde{\omega}] = 0 \in H^{n+1}(\widetilde{C}^{\mathfrak{g}})$

Moreover, the respective moments are in one-to-one correspondence with potentials of the respective cohomology classes.

This theorem recovers the following result

Corollary 4.5.7. *[29, Prop. 7.3, Theorem 7.4]* $\mathfrak{g}$*-equivariant weak moment maps are unique up to elements of* $\bigoplus_{k=1}^{n}\left(P^*_{k,g} \otimes \Omega^{n-k+1}_{cl}(M)\right)^{\mathfrak{g}}$*. In particular, if these groups vanish, then* $\mathfrak{g}$*-equivariant weak moment maps are unique.*

Remark 4.5.8. We end this section by noting that in symplectic geometry, considering actions of connected Lie groups, the conditions for being a homotopy moment map and an equivariant moment map coincide, i.e., a symplectic homotopy moment map is automatically equivariant.

Chapter 5

Lie 2-algebra moment maps

The material of this chapter is based on a paper ([44]) co-authored by the author of the thesis, with similar wording in many places.

5.1 Introduction

We saw in Chapter 3 that an n-plectic manifold (M, ω) is equipped with a Lie n-algebra of observables, which we denoted $L_\infty(M, \omega)$. A (homotopy) moment map in this case is defined as an L_∞-morphism $\{f_k\} : \mathfrak{g} \to L_\infty(M, \omega)$ such that $d(f_1(x)) = -\iota_{v_x}\omega$. We saw that (Theorem 3.4.13), under certain cohomological conditions on M, an action of $\mathfrak{g}$ by Hamiltonian vector fields admits a homotopy moment map if and only if a certain class in $H^{n+1}(\mathfrak{g})$ vanishes. If this class doesn't vanish, we saw that there is an L_∞-morphism between a central n-extension of $\mathfrak{g}$ and $L_\infty(M, \omega)$ (Proposition 3.4.14). The central n-extension is a Lie n-algebra, so this motivates us to define and investigate Lie n-algebra moment maps, i.e., morphisms between Lie n-algebras that are compatible with the action of $\mathfrak{g}$ on (M, ω). Another reason why considering Lie n-algebra moment maps is natural is that since multisymplectic geometry already forces us to consider an L_∞-morphism $\mathfrak{g} \to L_\infty(M, \omega)$, there is no reason to restrict ourselves to morphisms from Lie algebras, but rather allow more general objects on the left-hand side.

In this chapter we consider Lie 2-algebra moment maps and 2-plectic manifolds for simplicity, but it is reasonable to expect that similar results can be stated for Lie n-algebras and n-plectic manifolds. We characterize moment maps for a Lie 2-algebra $\mathfrak{h} \oplus \mathfrak{g}$ in cohomological terms, consecutively investigating existence and uniqueness in terms of 3 cochain complexes: the first one being the tensor product of the Chevalley-Eilenberg complex of $\mathfrak{h} \oplus \mathfrak{g}$ and the de Rham complex of M, the second one being the Chevalley-Eilenberg complex of $\mathfrak{h} \oplus \mathfrak{g}$ (i.e., a smaller complex than the first one), and finally the last one being the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g}$ (i.e., the "easiest" complex to deal with). This chapter is based on [44].

5.2 Homotopy moment maps for Lie 2-algebras

In this chapter, we assume the following set-up:

(M, ω) is a 2-plectic manifold,
$\mathfrak{g}$ is a Lie algebra,
$\mathfrak{g} \to \mathfrak{X}(M), x \mapsto v_x$ is a Lie algebra morphism, such that the v_x are Hamiltonian vector fields.

We extend Definition 3.4.1. Let $\mathfrak{h} \oplus \mathfrak{g}$ be a Lie 2-algebra, whose binary bracket extends the given Lie algebra structure on $\mathfrak{g}$. Recall from Definition 3.2.12 that a Lie 2-algebra is an L_∞-algebra concentrated in degrees 0 and -1, thus we consider $\mathfrak{h}$ to be in degree -1, and $\mathfrak{g}$ in degree 0. We denote the multibrackets l_1, l_2 and l_3 of this Lie 2-algebra by δ, $[\ ,\ ]$ and $[\ ,\ ,\ ]$, respectively.

Definition 5.2.1. A *homotopy moment map* for the Lie 2-algebra $\mathfrak{h}\oplus\mathfrak{g}$ (or $\mathfrak{h}\oplus\mathfrak{g}$ *moment map* for short) is an L_∞-morphism (f_1, f_2) from $(\mathfrak{h} \oplus \mathfrak{g}, \delta, [\ ,\ ], [\ ,\ ,\ ])$ to $(L_\infty(M, \omega), d, [\ ,\ ]', [\ ,\ ,\ ]')$ such that for all $x \in \mathfrak{g}$

$$-\iota_{v_x}\omega = d(f_1(x)).$$

Remark 5.2.2. Using Theorem 3.3.29 and Definition 3.2.20, we can see that explicitly, this means that the components

$$f_1 : \mathfrak{g} \to \Omega^1_{\mathrm{Ham}}(M),$$

$$f_1 : \mathfrak{h} \to C^\infty(M),$$

$$f_2 : \wedge^2\mathfrak{g} \to C^\infty(M),$$

satisfy the following equations for all $x, y, z \in \mathfrak{g}$ and $h \in \mathfrak{h}$ ([51, Def. 6.2], see also [67, Def. 5.3]):

$$d \circ f_1 = f_1 \circ \delta \tag{5.1}$$

$$d(f_2(x,y)) = f_1[x,y] - [f_1(x), f_1(y)]'$$

$$f_2(\delta h, x) = f_1[h,x],$$

$$f_1[x,y,z] - [f_1(x), f_1(y), f_1(z)]' = f_2(x,[y,z]) - f_2(y,[x,z]) + f_2(z,[x,y]).$$

Note that the image of $\delta\colon \mathfrak{h} \to \mathfrak{g}$ lies in the kernel of the infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M)$. Indeed, from equation (5.1) it follows that the Hamiltonian 1-form $f_1(\delta(h))$ is exact, implying the vanishing of its Hamiltonian vector field $v_{f_1(\delta(h))}$, which is the infinitesimal generator of the action corresponding to $\delta(h)$. As a consequence, if the infinitesimal $\mathfrak{g}$-action is effective (in the sense that the Lie algebra morphism $\mathfrak{g} \to \mathfrak{X}(M)$ has trivial kernel) then the unary bracket δ of $\mathfrak{h} \oplus \mathfrak{g}$ necessarily vanishes.

Hence, from now on in this article we assume the following (recall that a L_∞-algebra is called *minimal* if it has vanishing unary map):

the Lie 2-algebra $\mathfrak{h} \oplus \mathfrak{g}$ is minimal.

Remark 5.2.3. Any Lie n-algebra is L_∞-quasi-isomorphic to a minimal Lie n-algebra (see [16, §7] up to and including Cor. 7.5). Hence, this assumption does not imply any loss of generality. We thank Chris Rogers for pointing this out to us.

Such a Lie 2-algebra admits a simple well-known description that we now recall (see [3, Thm. 55] for more details).

Lemma 5.2.4. *A minimal Lie 2-algebra corresponds to the following data:*

- *a Lie algebra* $\mathfrak{g}$*,*
- *a* $\mathfrak{g}$*-representation* $\mathfrak{h}$*,*
- *a 3-cocycle c for this representation.*

The representation of $\mathfrak{g}$ *on* $\mathfrak{h}$ *is given by the binary bracket, and the cocycle for this representation is given by the ternary bracket.*

Proof. Let $\mathfrak{h}\oplus\mathfrak{g}$ be a minimal Lie 2-algebra. The higher Jacobi identities reduce to the following, for $h \in \mathfrak{h}$ and $x, y, z, u \in \mathfrak{g}$:

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0 \tag{5.2}$$

$$[[x,y],h]-[x,[y,h]]-[[x,h],y]=0 \tag{5.3}$$

$$\begin{aligned}[x,[y,z,u]]-[y,[x,z,u]]+[z,[x,y,u]]-[u,[x,y,z]] &= \\ &[[x,y],z,u]+[y,[x,z],u] \\ &[y,z,[x,u]]-[x,[y,z],u] \\ &-[x,z,[y,u]]+[x,y,[z,u]]\end{aligned} \tag{5.4}$$

The degree 0 component $\mathfrak{g}$ is a Lie algebra due to (5.2). That $\mathfrak{h}$ is a representation of $\mathfrak{g}$ follows from the fact that $[\mathfrak{g},\mathfrak{h}]$ lands in $\mathfrak{h}$, and from (5.3). That the ternary bracket $[\ ,\ ,\ ] =: c : \wedge^3\mathfrak{g} \to \mathfrak{h}$ is a 3-cocycle in the Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in $\mathfrak{h}$ follows from (5.4):

$$\begin{aligned}(d_{\mathfrak{g}}c)(x,y,z,u) &= x\cdot c(y,z,u)-y\cdot c(x,z,u)+z\cdot c(x,y,u)-u\cdot c(x,y,z) \\ &\quad - c([x,y],z,u)+c([x,z],y,u)-c([x,u],y,z) \\ &\quad - c([y,z],x,u)+c([y,u],x,z)-c([z,u],x,y)=0.\end{aligned}$$

□

5.3 A cohomological characterization of Lie 2-algebra moment maps

We saw in §3.4.4 that homotopy moment maps for Lie algebra actions can be characterized in terms of a certain double complex. In this section we obtain an analogous result for Lie 2-algebras.

Let $\mathfrak{h}\oplus\mathfrak{g}$ be a minimal Lie 2-algebra, let ω be a 2-plectic form on a manifold M, and let $\mathfrak{g}\to\mathfrak{X}(M), x\mapsto v_x$ be a Lie algebra morphism taking values in Hamiltonian vector fields.

5.3.1 A cohomological characterization

In analogy with §3.4.4, we introduce a double complex that is the tensor product of the Chevalley-Eilenberg complex of the Lie 2-algebra $\mathfrak{h} \oplus \mathfrak{g}$ and the de Rham complex of the manifold M. Note that we denote the total complex of this complex (C, d_{tot}), just as we denoted the total complex of the double complex (3.31) in §3.4.4, but there is an important difference: the de Rham complex of M is tensored with the Chevalley-Eilenberg complex of the Lie 2-algebra $\mathfrak{h} \oplus \mathfrak{g}$, not just the Lie algebra $\mathfrak{g}$, i.e., the (C, d_{tot}) in this chapter is different from the one in §3.4.4.

Let $(CE(L), d_{CE(L)})$ be the Chevalley-Eilenberg complex of the Lie 2-algebra $L = \mathfrak{h} \oplus \mathfrak{g}$ (see Example 3.2.17). Consider the double complex

$$C := CE(L) \otimes \Omega(M),$$

where $(\Omega(M), d)$ is the de Rham complex of M. We denote the resulting total complex by (C, d_{tot}), where

$$d_{tot} := d_{CE(L)} \otimes 1 + 1 \otimes d.$$

Define

$$\omega_k : \wedge^k \mathfrak{g} \to \Omega^{3-k}(M), (x_1, ..., x_k) \mapsto \iota(v_{x_1} \wedge ... \wedge v_{x_k})\omega,$$

as in (3.33), and

$$\widetilde{\omega} := \sum_{k=1}^{3} (-1)^{k-1} \omega_k,$$

as in (3.34).

Note that $\widetilde{\omega}$ is a degree 3 element of $CE(L) \otimes \Omega(M) = C$, using the canonical identification $\wedge \mathfrak{g}^* \cong S(s\mathfrak{g})^*$, where s denotes the degree shift defined in (3.3). The following lemma is analogous to Lemma 3.4.17.

Lemma 5.3.1. *$\widetilde{\omega}$ is d_{tot}-closed.*

Proof. It was shown in Lemma 3.4.17 that $\widetilde{\omega} \in S^{\bullet \geq 1}(s\mathfrak{g})^* \otimes \Omega(M)$ is closed with respect to the differential $D := d_{\mathfrak{g}} \otimes 1 + 1 \otimes d$, where $d_{\mathfrak{g}}$ is the Chevalley-Eilenberg differential of $\mathfrak{g}$.

The inclusion $j : (S^{\bullet \geq 1}(s\mathfrak{g})^*, d_{\mathfrak{g}}) \to (S^{\bullet \geq 1}(s(\mathfrak{h} \oplus \mathfrak{g}))^*, d_{CE(L)}) = (CE(L), d_{CE(L)})$ is a chain map, i.e., $j(d_{\mathfrak{g}}(\xi)) = d_{CE(L)}(j(\xi))$ for all $\xi \in S^k(s\mathfrak{g})^*$. This follows from:

1) $d_{CE(L)} = -d_2 + d_3$, because the unary bracket of $\mathfrak{h} \oplus \mathfrak{g}$ vanishes,

2) $-d_2 = d_{\mathfrak{g}}$ on elements of $S^k(s\mathfrak{g})^*$,

3) $d_3(\xi) = 0$, because the ternary bracket of $\mathfrak{h} \oplus \mathfrak{g}$ takes values in $\mathfrak{h}$,

where d_2 and d_3 are defined as in Example 3.2.17.

Thus, the inclusion $(S^{\bullet \geq 1}(s\mathfrak{g})^* \otimes \Omega(M), D) \to (CE(L) \otimes \Omega(M), d_{tot})$ is also a chain map, which means that $\widetilde{\omega} \in CE(L) \otimes \Omega(M)$ is also closed with respect to d_{tot}. □

Analogously to Proposition 3.4.18 we obtain:

Proposition 5.3.2. *There is a bijection*

$$\{\textit{moment maps for } \mathfrak{h} \oplus \mathfrak{g}\} \cong \{\mu \in C^2 : d_{tot}\mu = \widetilde{\omega}\}.$$

More precisely, for all[1]

$$\mu = \mu_1|_{\mathfrak{g}} + \mu_1|_{\mathfrak{h}} + \mu_2 \in C^2 = (s\mathfrak{g})^* \otimes \Omega^1(M) \ \oplus \ ((s\mathfrak{h})^* \oplus S^2(s\mathfrak{g})^*) \otimes C^\infty(M)$$

we have: $d_{tot}\mu = \widetilde{\omega}$ *iff*

$$\mu_1|_{\mathfrak{g}} : \mathfrak{g} \to \Omega^1_{Ham}$$

$$\mu_1|_{\mathfrak{h}} : \mathfrak{h} \to C^\infty(M)$$

$$\mu_2 : \wedge^2 \mathfrak{g} \to C^\infty(M)$$

are the components of a $\mathfrak{h} \oplus \mathfrak{g}$ *moment map.*

Note that, by Proposition 5.3.2, the set of moment maps for $\mathfrak{h} \oplus \mathfrak{g}$ forms an affine space.

Proof. Recall that a moment map $\phi = (\phi_1|_{\mathfrak{g}}, \phi_1|_{\mathfrak{h}}, \phi_2)$ for $(\mathfrak{h} \oplus \mathfrak{g}, [\ ,\ ], [\ ,\ ,\ ])$ has to satisfy the following equalities for $x, y, z \in \mathfrak{g}, h \in \mathfrak{h}$, by Remark 5.2.2

$$d(\phi_1|_{\mathfrak{g}}(x)) = -\iota_{v_x}\omega \tag{5.5}$$

$$d(\phi_1|_{\mathfrak{h}}(h)) = 0 \tag{5.6}$$

$$d(\phi_2(x,y)) = \phi_1|_{\mathfrak{g}}[x,y] - [\phi_1|_{\mathfrak{g}}(x), \phi_1|_{\mathfrak{g}}(y)]' \tag{5.7}$$

$$\phi_1|_{\mathfrak{h}}[h,x] = 0 \tag{5.8}$$

$$\phi_1|_{\mathfrak{h}}([x,y,z]) - [\phi_1|_{\mathfrak{g}}(x), \phi_1|_{\mathfrak{g}}(y), \phi_1|_{\mathfrak{g}}(z)]' = \phi_2(x,[y,z]) - \phi_2(y,[x,z]) + \phi_2(z,[x,y]), \tag{5.9}$$

[1] Here elements of $(s\mathfrak{h})^*$ have degree two, since elements of $\mathfrak{h}$ have degree -1.

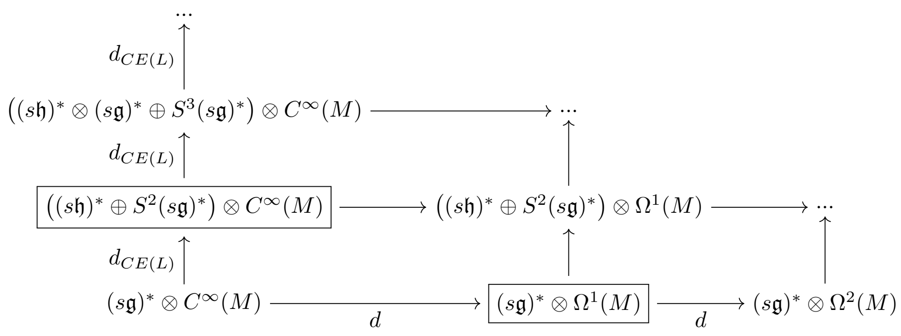


Figure 5.1: The double complex $C = CE(L) \otimes \Omega(M)$, with highlighted components of total degree 2, i.e., C^2. Recall that elements of $(s\mathfrak{h})^*$ have degree 2.

where $[\ ,\ ]'$ and $[\ ,\ ,\ ]'$ are the binary and the ternary bracket of $L_\infty(M, \omega)$.

Now let $\mu \in C^2$. Comparing the components of $d_{tot}\mu \in C^3$ and $\widetilde{\omega} \in C^3$ with the same bi-degree, we see that the equation $d_{tot}\mu = \widetilde{\omega}$ is equivalent to the following five equations:

$$(1 \otimes d)(\mu_1|_{\mathfrak{g}}) = \omega_1$$

$$(1 \otimes d)(\mu_1|_{\mathfrak{h}}) = 0$$

$$-(d_2 \otimes 1)(\mu_1|_{\mathfrak{g}}) + (1 \otimes d)\mu_2 = -\omega_2$$

$$-(d_2 \otimes 1)(\mu_1|_{\mathfrak{h}}) = 0$$

$$(d_3 \otimes 1)(\mu_1|_{\mathfrak{h}}) - (d_2 \otimes 1)\mu_2 = \omega_3$$

Rewriting these equations in terms of the respective Lie 2-algebra brackets and evaluating on $x, y, z \in \mathfrak{g}, h \in \mathfrak{h}$, we get the following equations:

$$d(\mu_1|_{\mathfrak{g}}(x)) = -\iota_{v_x}\omega \tag{5.10}$$

$$d(\mu_1|_{\mathfrak{h}}(h)) = 0 \tag{5.11}$$

$$-\mu_1|_{\mathfrak{g}}([x, y]) + d(\mu_2(x, y)) = -[\mu_1|_{\mathfrak{g}}(x), \mu_1|_{\mathfrak{g}}(y)]' \tag{5.12}$$

$$\mu_1|_{\mathfrak{h}}([h, x]) = 0 \tag{5.13}$$

$$-\mu_1|_{\mathfrak{h}}([x, y, z]) - \mu_2([x, y], z) + \mu_2([x, z], y) - \mu_2([y, z], x) = -[\mu_1|_{\mathfrak{g}}(x), \mu_1|_{\mathfrak{g}}(y), \mu_1|_{\mathfrak{g}}(z)]' \tag{5.14}$$

Comparing equations (5.5)-(5.9) and (5.10)-(5.14), we see that $(\mu_1|_{\mathfrak{g}}, \mu_1|_{\mathfrak{h}}, \mu_2)$ is a moment map iff $d_{tot}\mu = \widetilde{\omega}$. □

5.4 Existence results and a construction

We use the characterization of moment maps for Lie 2-algebras given in §5.3.1 to obtain existence results and construct explicit examples.

Again, let $\mathfrak{h} \oplus \mathfrak{g}$ be a minimal Lie 2-algebra, let ω be a 2-plectic form on a manifold M, and let $\mathfrak{g} \to \mathfrak{X}(M), x \mapsto v_x$ be a Lie algebra morphism taking values in Hamiltonian vector fields.

5.4.1 Two immediate existence results

Given a minimal Lie 2-algebra $\mathfrak{h} \oplus \mathfrak{g}$, the projection $\mathfrak{h} \oplus \mathfrak{g} \to \mathfrak{g}$ is a strict L_∞-morphism; hence, we immediately have:

Corollary 5.4.1. *From a $\mathfrak{g}$ moment map, by composition with the above projection, one obtains a $\mathfrak{h} \oplus \mathfrak{g}$ moment map.*

Remark 5.4.2. The inclusion $\mathfrak{g} \to \mathfrak{h} \oplus \mathfrak{g}$, however, is a not a strict L_∞-morphism in general: it is iff $\mathfrak{h} \oplus \mathfrak{g}$ is a differential graded Lie algebra (see Example 3.2.11). In particular, moment maps for a differential graded Lie algebra $\mathfrak{h} \oplus \mathfrak{g}$ exist iff $\mathfrak{g}$ moment maps exist.

Now fix $p \in M$. By Theorem 3.4.13, the existence of a $\mathfrak{g}$ moment map is equivalent to $[\omega_{3p}]_{\mathfrak{g}} = 0$, where ω_{3p} is defined by

$$\omega_{3p}(x, y, z) = \omega(v_x, v_y, v_z)|_p \tag{5.15}$$

for all $x, y, z \in \mathfrak{g}$

When $[\omega_{3p}]_{\mathfrak{g}} \neq 0$, there exists no moment map for $\mathfrak{g}$, but, when M is connected and $H^1(M) = 0$, there always exists a moment map for the central 2-extension (see Proposition 3.4.14). Indeed, denote by

$$\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$$

the Lie 2-algebra with the underlying vector space $\mathbb{R} \oplus \mathfrak{g}$ and the multibrackets $[x, y] := [x, y]_{\mathfrak{g}}$ for $x, y \in \mathfrak{g}$, ternary bracket equal to $-\omega_{3p}$, and all other brackets being trivial. We paraphrase a special case of Proposition 3.4.14:

Proposition 5.4.3. *If M is connected and $H^1(M) = 0$, then there exists a moment map for $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$.*

The construction of this moment map is as follows: let $\gamma_1 \colon \mathfrak{g} \to \Omega^1_{\mathrm{Ham}}(M)$ be a linear map such that $\gamma_1(x)$ is a Hamiltonian 1-form for the Hamiltonian vector field v_x, for all x. Then a moment map $\gamma = (\gamma_1, \gamma_2)$ is given by:

$$\gamma_1 \colon \mathfrak{g} \to \Omega^1_{\mathrm{Ham}}(M) \tag{5.16}$$

$$\gamma_1 \colon \mathbb{R} \to C^\infty(M), \quad r \mapsto r,$$

$$\gamma_2 \colon \wedge^2 \mathfrak{g} \to C^\infty(M) \tag{5.17}$$

where, for all $x, y \in \mathfrak{g}$, $\gamma_2(x, y)$ is the unique solution of the equation

$$\gamma_1([x_1, x_2]) - \iota(v_{x_1} \wedge v_{x_2})\omega = d(\gamma_2(x_1, x_2))$$

with $\gamma_2(x, y)_p = 0$.

From now on we fix a linear map $\gamma_1 \colon \mathfrak{g} \to \Omega^1_{Ham}(M)$ as above, and consequently a moment map γ for $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$.

5.4.2 A constructive existence result in terms of $H^3(CE(L))$.

By Proposition 5.3.2, moment maps for $\mathfrak{h} \oplus \mathfrak{g}$ exist iff the cohomology class of $\widetilde{\omega} \in C^3$ vanishes. In this subsection, which is inspired by [18, §5], we obtain existence results for $\mathfrak{h} \oplus \mathfrak{g}$ moment maps in terms of the cohomology of the Chevalley-Eilenberg complex $CE(L)$ of this Lie 2-algebra. Note that the latter is smaller than C, and thus more manageable. Furthermore, we give an explicit construction of $\mathfrak{h} \oplus \mathfrak{g}$ moment maps.

Fix a point $p \in M$. The map

$$r : (C, d_{tot}) \to (CE(L), d_{CE(L)})$$

$$\eta \otimes \alpha \mapsto \eta \cdot \alpha_p,$$

is a chain map, where $\alpha_p \in \mathbb{R}$ is declared to vanish if $\alpha \in \Omega^{\geq 1}(M)$. Since $\widetilde{\omega}$ is d_{tot}-closed by Lemma 3.1, its image $r(\widetilde{\omega}) = \omega_{3p} \in CE(L)^3$ is $d_{CE(L)}$-closed; hence, it defines a class $[\omega_{3p}]_{CE(L)}$ in $H^3(CE(L))$, the Chevalley-Eilenberg cohomology of $\mathfrak{h} \oplus \mathfrak{g}$.

Proposition 5.4.4. *If a $\mathfrak{h} \oplus \mathfrak{g}$ moment map exists, then $[\omega_{3p}]_{CE(L)} = 0$.*

Proof. By Proposition 3.2, if a moment map exists, then $[\widetilde{\omega}]_C = 0 \in H^3(C)$. Hence, we have $[\omega_{3p}]_{CE(L)} = [r]([\widetilde{\omega}]_C) = 0$, where $[r] : H^3(C) \to H^3(CE(L))$ is the induced map in cohomology. □

Conversely, we are now going to show that, if $[\omega_{3p}]_{CE(L)} = 0$ and certain cohomological assumptions on M are satisfied, there exists a moment map for $\mathfrak{h} \oplus \mathfrak{g}$. Our approach is constructive.

Remark 5.4.5. An alternative approach, which, however, requires stronger assumptions and is not constructive, is the following. Assume that $H^1(M) = H^2(M) = 0$. By the Künneth theorem, $H^3(C) \cong H^3(CE(L)) \otimes H^0(M) \cong H^3(CE(L))$, and the map $[r] : H^3(C) \to H^3(CE(L))$ induced in cohomology is an isomorphism. Thus, if $[\omega_{3p}]_{CE(L)} = 0$, then $[\widetilde{\omega}]_C = 0$, and, by Proposition 5.3.2, there exists a moment map.

In what follows, given an element η of

$$CE(L)^2 = (s\mathfrak{h})^* \oplus S^2(s\mathfrak{g})^*,$$

we will denote the first and the second component of η respectively by $\eta|_{s\mathfrak{h}}$ and $\eta|_{S^2 s\mathfrak{g}}$.

Lemma 5.4.6. *An element $\eta \in CE(L)^2$ satisfies $d_{CE(L)}\eta = \omega_{3p}$ iff*

$$\begin{cases} d_{\mathfrak{g}}(\eta|_{S^2 s\mathfrak{g}}) + d_3(\eta|_{s\mathfrak{h}}) = \omega_{3p} \\ d_2(\eta|_{s\mathfrak{h}}) = 0 \end{cases} \tag{5.18}$$

Proof. The claim follows easily by computing $d_{CE(L)}\eta$ and comparing the respective components of $d_{CE(L)}\eta$ and ω_{3p} in $CE(L)$. □

Remark 5.4.7. We will consider this system in detail in §5.5.1. For the moment we only remark that the second equation is equivalent to $\eta|_{s\mathfrak{h}}$ lying in the subspace $[\mathfrak{g}, \mathfrak{h}]^0 := \{\xi \in \mathfrak{h}^* : \xi(v) = 0,\ \forall v \in [\mathfrak{g}, \mathfrak{h}]\}$.

Lemma 5.4.8. *There is a bijection between*

$$\{\eta \in CE(L)^2 : d_{CE(L)}\eta = \omega_{3p}\}$$

and

$$\{L_\infty\text{-morphisms } f\colon \mathfrak{h} \oplus \mathfrak{g} \to \mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g} \text{ with } f_1|_{\mathfrak{g}} = Id_{\mathfrak{g}}\}.$$

The bijection maps $\eta = \eta|_{s\mathfrak{h}} + \eta|_{S^2 s\mathfrak{g}}$ to the L_∞-morphisms f with components $f_1 = (\eta|_{s\mathfrak{h}}, Id_{\mathfrak{g}})$ and $f_2 = \eta|_{S^2 s\mathfrak{g}}$.

Proof. We will denote the ternary bracket of $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$ by $[\ ,\ ,\ ]^{\diamond}$. For f to be an L_∞ morphism $f : \mathfrak{h} \oplus \mathfrak{g} \to \mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$, its components

$$f_1|_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$$

$$f_1|_{\mathfrak{h}} : \mathfrak{h} \to \mathbb{R}$$

$$f_2 : \wedge^2 \mathfrak{g} \to \mathbb{R}$$

must satisfy the following relations for $x, y, z \in \mathfrak{g}, h \in \mathfrak{h}$:

$$f_1|_{\mathfrak{h}}[h, x] = 0, \tag{5.19}$$

$$f_1|_{\mathfrak{g}}[x, y] - [f_1|_{\mathfrak{g}}(x), f_1|_{\mathfrak{g}}(y)]_{\mathfrak{g}} = 0$$

$$f_1|_{\mathfrak{h}}([x, y, z]) - [f_1|_{\mathfrak{g}}(x), f_1|_{\mathfrak{g}}(y), f_1|_{\mathfrak{g}}(z)]^{\diamond} = f_2(x, [y, z]) - f_2(y, [x, z]) + f_2(z, [x, y]). \tag{5.20}$$

These equalities follow from Definition 3.2.20 and are explicitly stated in [67, Def. 5.3] for arbitrary Lie 2-algebras.

Clearly (5.19) is equivalent to $d_2(f_1|_{\mathfrak{h}}) = 0$. When $f_1|_{\mathfrak{g}} = Id_{\mathfrak{g}}$, rewriting (5.20) using the definition of the bracket $[\ ,\ ,\ ]^{\diamond}$, we get:

$$f_1|_{\mathfrak{h}}([x, y, z]) - f_2(x, [y, z]) + f_2(y, [x, z]) - f_2(z, [x, y]) = -\omega_{3p}(x, y, z)$$

The latter equality can be written as $d_{\mathfrak{g}}(f_2) + d_3(f_1|_{\mathfrak{h}}) = \omega_{3p}$. Applying Lemma 5.4.6 concludes the proof. ☐

By composition, we immediately obtain the following result, which also provides an explicit construction for $\mathfrak{h} \oplus \mathfrak{g}$ moment maps.

Theorem 5.4.9. *Assume M is connected and $H^1(M) = 0$. Let $\eta \in CE(L)^2$ satisfy $d_{CE(L)}\eta = \omega_{3p}$. Then*

$$\phi^\eta := \gamma \circ f$$

is a moment map for $\mathfrak{h} \oplus \mathfrak{g}$, where f is constructed out of η as in Lemma 5.4.8, and γ is given just below Proposition 5.4.3.

Remark 5.4.10. i) Explicitly, ϕ^η is given as follows, where $x, y \in \mathfrak{g}, h \in \mathfrak{h}$:

$$\begin{aligned}
\phi_1^\eta(x) &= \gamma_1(x) \\
\phi_1^\eta(h) &= \gamma_1(\eta(h)) = \eta(h) \\
\phi_2^\eta(x, y) &= \gamma_1(f_2(x, y)) + \gamma_2(f_1(x), f_1(y)) \\
&= \eta(x, y) + \gamma_2(x, y).
\end{aligned}$$

ii) When $\mathfrak{h} = 0$, the moment map ϕ^η for $\mathfrak{g}$ agrees with the one constructed in [7, Prop. 9.6] and the one we sketched in the proof of Theorem 3.4.13. (Note that when $\mathfrak{h} = 0$, the choice of η amounts to a choice of primitive of ω_{3p} in the Chevalley-Eilenberg complex of $\mathfrak{g}$.)

iii) The $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$ moment map γ itself is obtained as in Theorem 5.4.9 from the solution $\eta(r) = r,\ \eta(x, y) \equiv 0,\ r \in \mathbb{R},\ x, y \in \mathfrak{g}$ of the equation $d_{CE(L)}\eta = \omega_{3p}$. Indeed, under the bijection of Lemma 5.4.8, η corresponds to the identity on $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$.

By combining the results of Proposition 5.4.4 and Theorem 5.4.9, we summarize as follows the existence results obtained in this subsection:

Corollary 5.4.11. *Let (M, ω) be a 2-plectic manifold, and $\mathfrak{g} \to \mathfrak{X}(M)$ a Lie algebra taking values in Hamiltonian vector fields. Let $\mathfrak{h} \oplus \mathfrak{g}$ be a minimal Lie 2-algebra. Fix $p \in M$.*

If a moment map for $\mathfrak{h} \oplus \mathfrak{g}$ exists, then $[\omega_{3p}]_{CE(L)} = 0$. The converse holds whenever $H^1(M) = 0$ and M is connected.

5.4.3 A uniqueness result

In this subsection we assume that $H^1(M) = 0$ and M is connected. In §5.4.2 we addressed the existence of moment maps for $\mathfrak{h} \oplus \mathfrak{g}$. Here we show that any moment map for $\mathfrak{h} \oplus \mathfrak{g}$ is cohomologous to one constructed by composition in Theorem 5.4.9.

Fix $p \in M$. Consider again the map introduced in §5.4.2:

$$r : (C, d_{tot}) \to (CE(L), d_{CE(L)})$$

$$\eta \otimes \alpha \mapsto \eta \cdot \alpha_p.$$

We remark that the map induced in cohomology in degree 2

$$[r] : H^2(C) \to H^2(CE(L))$$

is an isomorphism. This follows from the fact that, by the Künneth theorem,

$$H^2(C) = H^2(CE(L) \otimes \Omega(M)) \cong H^2(CE(L)) \otimes \mathbb{R},$$

where we used that $CE(L)$ is concentrated in positive degrees and $H^1(M) = 0$.

To any $\eta \in CE(L)^2$ with $d_{CE(L)}\eta = \omega_{3p}$ in Theorem 5.4.9 we associated a $\mathfrak{h} \oplus \mathfrak{g}$ moment map ϕ^η, which we now view as an element of C^2.

Lemma 5.4.12. *For all η as above: $r(\phi^\eta) = \eta$*

Proof. Using Remark 5.4.10 i) we find:

$$\phi_1^\eta|_{\mathfrak{g}} = \gamma_1|_{\mathfrak{g}} \qquad \in (s\mathfrak{g})^* \otimes \Omega^1(M)$$

$$\phi_1^\eta|_{\mathfrak{h}} = \eta|_{s\mathfrak{h}} \otimes 1 \qquad \in (s\mathfrak{h})^* \otimes C^\infty(M)$$

$$\phi_2^\eta = (\eta|_{S^2 s\mathfrak{g}} \otimes 1 + \gamma_2) \qquad \in S^2(s\mathfrak{g})^* \otimes C^\infty(M)$$

Now note that $r(\phi_1^\eta|_{\mathfrak{g}}) = 0$ by definition of the map r, and $r(\phi_1^\eta|_{\mathfrak{h}}) = \eta|_{s\mathfrak{h}}$.

Finally $r(\phi_2^\eta) = \eta|_{S^2 s\mathfrak{g}} + r(\gamma_2) = \eta|_{S^2 s\mathfrak{g}}$. Indeed $r(\gamma_2) = \gamma_{2p} = 0$, because $\gamma_2(x, y)$ vanishes at point p for any $x, y \in \mathfrak{g}$ by construction (see 5.17). Hence $r(\phi^\eta) = \eta|_{s\mathfrak{h}} + \eta|_{S^2 s\mathfrak{g}} = \eta$. □

The difference of any two moment maps is a closed element of C^2, by Proposition 5.3.2. Extending [18, Rem. 7.10] we define:

Definition 5.4.13. Two moment maps $\mu, \mu' \in C^2$ are called *inner equivalent* if $\mu - \mu' = d_{tot}\alpha$ for some $\alpha \in C^1$.

The following proposition gives conditions to ensure that all moment maps are inner equivalent to those constructed earlier.

Proposition 5.4.14. *Let M be a manifold with $H^1(M) = 0$. If $\mu \in C^2$ is a moment map for $\mathfrak{h} \oplus \mathfrak{g}$, then μ and $\phi^{r(\mu)}$ are inner equivalent.*

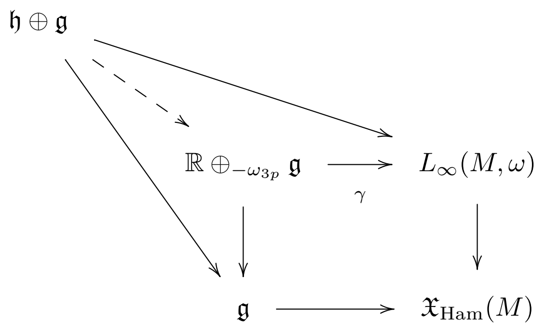
Remark 5.4.15. Note that, since r is a chain map, and by Proposition 5.3.2, if μ is a moment map, then $r(\mu)$ is a solution of $d_{CE(L)}\eta = \omega_{3p}$. Hence $\phi^{r(\mu)}$ is indeed well-defined.

Remark 5.4.16. In §5.4 we fixed a choice of linear map $\gamma_1 \colon \mathfrak{g} \to \Omega^1_{\mathrm{Ham}}(M)$ providing Hamiltonian 1-forms for the generators of the action. For any $\eta \in CE(L)^2$ satisfying $d\eta = \omega_{3p}$, the moment map ϕ^η constructed in Theorem 5.4.9 depends on this choice. However, making a different choice for γ_1 delivers a moment map that is inner equivalent to ϕ^η. This can be seen using Lemma 5.4.12 and Proposition 5.4.14, or also by a direct computation.

Proof. We have $r(\mu - \phi^{r(\mu)}) = r(\mu) - r(\phi^{r(\mu)}) = 0$ by Lemma 5.4.12. In particular, for the map $[r]$ induced in cohomology, we have $[r][\mu - \phi^{r(\mu)}] = 0$. But the map $[r]$ is an isomorphism in degree 2, hence the cohomology class $[\mu - \phi^{r(\mu)}]$ in $H^2(C)$ also vanishes, i.e., $\mu - \phi^{r(\mu)} = d_{tot}\alpha$ for some $\alpha \in C^1$. □

Proposition 5.4.14 immediately implies:

Corollary 5.4.17. *The Lie 2-algebra* $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$ *is* universal *in the following sense: provided* $H^1(M) = 0$*, any moment map for a Lie 2-algebra* $\mathfrak{h} \oplus \mathfrak{g}$ *is inner equivalent to one that factors through* $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$.



5.5 Revisiting the existence results

As earlier, let $\mathfrak{h} \oplus \mathfrak{g}$ be a minimal Lie 2-algebra, let ω be a 2-plectic form on a manifold M, and let $\mathfrak{g} \to \mathfrak{X}(M), x \mapsto v_x$ be a Lie algebra morphism taking values in Hamiltonian vector fields.

An answer to the existence question for $\mathfrak{h} \oplus \mathfrak{g}$ moment maps was given in Corollary 5.4.11 in terms of the cohomology of the Chevalley-Eilenberg complex $CE(L)$ of the Lie 2-algebra. However, the latter complex is quite large and involved. In this section we rephrase that answer in two ways: one that is explicit and easily applicable to examples (§5.5.1), and one that is phrased directly in terms of the Lie 2-algebra (Proposition 5.5.11) rather than in terms of its constituents (as in Lemma 5.2.4).

5.5.1 An explicit characterization of existence in terms of $H^3(\mathfrak{g})$

In this subsection we answer the question of existence of $\mathfrak{h} \oplus \mathfrak{g}$ moment maps in terms of the familiar Lie algebra cohomology of $\mathfrak{g}$.

Corollary 5.4.11 expresses the existence of a moment map for $\mathfrak{h} \oplus \mathfrak{g}$ in terms of the vanishing of $[\omega_{3p}]_{CE(L)}$. Recall that the latter condition is equivalent to the existence of a solution of the system (5.18). We now formulate the first equation of the system 5.18 in terms of the Lie algebra cohomology of $\mathfrak{g}$.

Lemma 5.5.1. *For any $\xi \in [\mathfrak{g},\mathfrak{h}]^0 \subset \mathfrak{h}^*$, where $[\mathfrak{g},\mathfrak{h}]^0$ denotes the annihilator of $[\mathfrak{g},\mathfrak{h}]$, the element $d_3\xi \in \wedge^3\mathfrak{g}^*$ is $d_\mathfrak{g}$-closed.*

Proof. Recall that d_3 was defined in Example 3.2.17. For all $x,y,z,u \in \mathfrak{g}$ we compute

$$\begin{aligned} d_\mathfrak{g}(d_3\xi)(x,y,z,u) &= -(d_3\xi)([x,y],z,u) + (d_3\xi)([x,z],y,u) - (d_3\xi)([x,u],y,z) \\ &\quad - (d_3\xi)([y,z],x,u) + (d_3\xi)([y,u],x,z) - (d_3\xi)([z,u],x,y) \\ &= \xi(c([x,y],z,u)) - \xi(c([x,z],y,u)) + \xi(c([x,u],y,z)) \\ &\quad + \xi(c([y,z],x,u)) - \xi(c([y,u],x,z)) + \xi(c([z,u],x,y)) \\ &= \xi\big(x\cdot c(y,z,u) - y\cdot c(x,z,u) + z\cdot c(x,y,u) - u\cdot c(x,y,z)\big) \\ &= 0, \end{aligned}$$

where in the third equality we used that c (given by the ternary bracket of $\mathfrak{h}\oplus\mathfrak{g}$, as in Lemma 5.2.4) is a 3-cocycle in the Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in the representation $\mathfrak{h}$, and in the last equality the condition $\xi \in [\mathfrak{g},\mathfrak{h}]^0$. □

Hence, we can consider the linear map

$$\begin{aligned} \Psi : [\mathfrak{g},\mathfrak{h}]^0 &\to H^3(\mathfrak{g}) \\ \xi &\mapsto [d_3\xi]_\mathfrak{g} = -[\xi\circ c]_\mathfrak{g}. \end{aligned} \tag{5.21}$$

to the third Lie algebra cohomology group of $\mathfrak{g}$.

Lemma 5.5.2. *i) Let $\eta \in (s\mathfrak{h})^* \oplus S^2(s\mathfrak{g})^*$ satisfy $d_{CE(L)}\eta = \omega_{3p}$. Then $\eta|_{s\mathfrak{h}}$ lies in the preimage of $[\omega_{3p}]_\mathfrak{g}$ under Ψ.*

ii) Conversely, let $\xi \in [\mathfrak{g},\mathfrak{h}]^0$ lie in the preimage of $[\omega_{3p}]_\mathfrak{g}$ under Ψ. Then we can find $\phi \in S^2 s\mathfrak{g}$ so that $\xi+\phi$ satisfies $d_{CE(L)}(\xi+\phi) = \omega_{3p}$.

Proof. If η is a solution to the system (5.18), then the second equation of the system says that $\eta|_{s\mathfrak{h}} \in [\mathfrak{g},\mathfrak{h}]^0$ (see Remark 5.4.7), and, taking cohomology classes in the first equation, we see that this element is mapped by Ψ to $[\omega_{3p}]_\mathfrak{g}$. The converse is proved by reversing the argument. □

The above lemma immediately implies:

Proposition 5.5.3. $[\omega_{3p}]_{CE(L)} = 0$ *iff* $[\omega_{3p}]_{\mathfrak{g}}$ *lies in the image of* Ψ*.*

We now give an alternative characterization of the map Ψ. Since $[\mathfrak{g},\mathfrak{h}]$ is a subrepresentation of $\mathfrak{h}$, we can look at the quotient representation $\mathfrak{h}/[\mathfrak{g},\mathfrak{h}]$, which is a trivial representation. Define $c_{red} := pr \circ c : \wedge^3\mathfrak{g} \to \mathfrak{h}/[\mathfrak{g},\mathfrak{h}]$. This map is a cocycle in the Lie algebra cohomology of $\mathfrak{g}$ with values in $\mathfrak{h}/[\mathfrak{g},\mathfrak{h}]$, as a consequence of the facts that c is a cocycle and the quotient map $\mathfrak{h} \to \mathfrak{h}/[\mathfrak{g},\mathfrak{h}]$ is a morphism of representations. Thus

- the Lie algebra $\mathfrak{g}$,
- the trivial $\mathfrak{g}$-representation $\mathfrak{h}/[\mathfrak{g},\mathfrak{h}]$,
- the 3-cocycle $c_{red} = pr \circ c$

define a minimal Lie 2-algebra. Its underlying graded vector space is $(\mathfrak{h}/[\mathfrak{g},\mathfrak{h}])\oplus\mathfrak{g}$, with $\mathfrak{h}/[\mathfrak{g},\mathfrak{h}]$ in degree -1, and $\mathfrak{g}$ in degree 0, and we will refer to this Lie 2-algebra as the *reduced Lie 2-algebra* corresponding to $\mathfrak{h}\oplus\mathfrak{g}$.

We can rewrite the map Ψ as follows, using the reduced Lie 2-algebra:

$$\begin{aligned}\Psi : (\mathfrak{h}/[\mathfrak{g},\mathfrak{h}])^* &\to H^3(\mathfrak{g}) \\ \widetilde{\xi} &\mapsto -[\widetilde{\xi}\circ c_{red}]_{\mathfrak{g}}\end{aligned} \tag{5.22}$$

In other words, Ψ maps $\widetilde{\xi}$ to the $\widetilde{\xi}$-component of[2] $[c_{red}]_{\mathfrak{g}} \in H^3(\mathfrak{g},\mathfrak{h}/[\mathfrak{g},\mathfrak{h}]) \cong H^3(\mathfrak{g})\otimes\mathfrak{h}/[\mathfrak{g},\mathfrak{h}]$. Hence, we can rephrase Proposition 5.5.3 by saying that $[\omega_{3p}]_{CE(L)} = 0$ iff $[c_{red}]_{\mathfrak{g}}$ has a component equal to $[\omega_{3p}]_{\mathfrak{g}}$.

Remark 5.5.4. Suppose the representation $\mathfrak{h}$ is a completely reducible representation (this happens, for instance, when $\mathfrak{g}$ is semisimple or is integrated by a compact Lie group), i.e., $\mathfrak{h} = \oplus_{i=1}^m\mathfrak{h}_i$ is a direct sum of irreducible subrepresentations $\mathfrak{h}_i$. Note that for every i, either $[\mathfrak{g},\mathfrak{h}_i] = \mathfrak{h}_i$ or $\mathfrak{h}_i$ is the trivial 1-dimensional representation. We may reorder the indices so that the trivial 1-dimensional subrepresentations (if any) are exactly $\mathfrak{h}_1,\dots,\mathfrak{h}_k$ for some $k \leq m$. Then $\mathfrak{h}_{red} = \oplus_{i=1}^k\mathfrak{h}_i$ consists of these trivial sub-representations. Decomposing into components an $\mathfrak{h}$-valued 3-cocycle $c \in \wedge^3\mathfrak{g}^*\otimes\mathfrak{h}$, we obtain an $\mathfrak{h}_i$-valued 3-cocycle c_i for every i, and c_{red} has components $c_1,\dots,c_k$. Hence $[\omega_{3p}]_{\mathfrak{g}}$ lies in the image of Ψ iff some linear combination of $[c_1],\dots,[c_k]$ equals $[\omega_{3p}]_{\mathfrak{g}}$.

[2]The isomorphism holds since $\mathfrak{h}/[\mathfrak{g},\mathfrak{h}]$ is a trivial representation of $\mathfrak{g}$.

We now apply Proposition 5.5.3 to obtain existence statements and obstructions for $\mathfrak{h}\oplus\mathfrak{g}$ moment maps. The case $[\omega_{3p}]_{\mathfrak{g}} = 0$ is not interesting, at least if $H^1(M)$ vanishes, since then a $\mathfrak{g}$ moment map exists (Theorem 3.4.13), and thus a $\mathfrak{h}\oplus\mathfrak{g}$ moment map exist too (Corollary 5.4.1). Hence, now we focus on the case $[\omega_{3p}]_{\mathfrak{g}} \neq 0$.

Proposition 5.5.5. *Assume $[\omega_{3p}]_{\mathfrak{g}} \neq 0$. If $[c_{red}]_{\mathfrak{g}} = 0 \in H^3(\mathfrak{g}, \mathfrak{h}/[\mathfrak{g},\mathfrak{h}])$, then there exists no $\mathfrak{h}\oplus\mathfrak{g}$ moment map.*

Proof. The characterization (5.22) of Ψ makes clear that $[c_{red}]_{\mathfrak{g}} = 0$ iff Ψ is the zero map. By Proposition 5.5.3 we have $[\omega_{3p}]_{CE(L)} \neq 0$. We conclude, using Proposition 5.4.4. □

Note that the assumption $[c_{red}]_{\mathfrak{g}} = 0$ is implied by either of the following conditions:

- The representation $\mathfrak{h}$ of $\mathfrak{g}$ satisfies $[\mathfrak{g},\mathfrak{h}] = \mathfrak{h}$, for in that case $\mathfrak{h}/[\mathfrak{g},\mathfrak{h}] = 0$.
- The cocycle c satisfies[3] $[c] = 0$, because the quotient map $\mathfrak{h} \to \mathfrak{h}/[\mathfrak{g},\mathfrak{h}]$ is a morphism of representations and the induced map $H(\mathfrak{g},\mathfrak{h}) \to H(\mathfrak{g}, \mathfrak{h}/[\mathfrak{g},\mathfrak{h}])$ maps $[c]$ to $[c_{red}]$.

Remark 5.5.6. Assuming $[\omega_{3p}]_{\mathfrak{g}} \neq 0$, if there exists a $\mathfrak{h}\oplus\mathfrak{g}$ moment map, then the representation must satisfy $[\mathfrak{g},\mathfrak{h}] \neq \mathfrak{h}$ (this follows from the first bullet point above). An easy general fact about Lie algebra representations then implies that either $\mathfrak{h}$ is not irreducible, or $\mathfrak{h}$ is the trivial 1-dimensional representation.

A positive result is the following:

Proposition 5.5.7. *Assume $[\omega_{3p}]_{\mathfrak{g}} \neq 0$. If $H^3(\mathfrak{g})$ is one-dimensional and $[c_{red}]_{\mathfrak{g}} \neq 0$, then $[\omega_{3p}]_{CE(L)} = 0$. Thus, if $H^1(M) = 0$, then there exists a moment map for $\mathfrak{h}\oplus\mathfrak{g}$.*

Proof. By the characterization 5.22, the map Ψ is surjective. Hence, the preimage of $[\omega_{3p}]_{\mathfrak{g}}$ under Ψ is nonempty. By Proposition 5.5.3, we have $[\omega_{3p}]_{CE(L)} = 0$. We finish, using Theorem 5.4.9. □

[3] In particular, this condition is satisfied when $c = 0$, i.e. $\mathfrak{h}\oplus\mathfrak{g}$ is a graded Lie algebra. In this case, the conclusion of Proposition 5.5.5 also follows from Remark 5.4.2 and Proposition 3.4.12.

5.5.2 An alternative characterization of existence

In this subsection we give an alternative answer to the existence question for $\mathfrak{h} \oplus \mathfrak{g}$ moment maps.

Lemma 5.5.8. *Let $\mathfrak{h} \oplus \mathfrak{g}$ be a minimal Lie 2-algebra. The following are equivalent:*

a) there is a surjective[4] L_∞-morphism from $\mathfrak{h} \oplus \mathfrak{g}$ to $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$ which is $Id_\mathfrak{g}$ on $\mathfrak{g}$,

b) there is a quotient of $\mathfrak{h} \oplus \mathfrak{g}$ which is L_∞-isomorphic[5] to $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$ by a morphism that is $Id_\mathfrak{g}$ on $\mathfrak{g}$.

Proof. $a) \Leftarrow b)$: just compose the morphism from $\mathfrak{h} \oplus \mathfrak{g}$ to its quotient with the L_∞-isomorphism from the latter to $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$.

$a) \Rightarrow b)$: let f be a morphism as in a), and denote $\xi := f_1|_\mathfrak{h}$. Then $s(\ker(\xi))$ is a L_∞-ideal of $\mathfrak{h} \oplus \mathfrak{g}$, as can be seen from (5.19). Furthermore, f descends to an L_∞-morphism from the quotient $(\mathfrak{h}/\ker(\xi)) \oplus \mathfrak{g}$ to $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$. The latter reads $Id_\mathfrak{g}$ on $\mathfrak{g}$, and is an L_∞-isomorphism because $\xi \neq 0$ due to the surjectivity of f. □

Remark 5.5.9. In the proof of "$a) \Rightarrow b)$" above, the quotient L_∞-algebra is strictly isomorphic to $\mathbb{R} \oplus_{\xi \circ c} \mathfrak{g}$ by the map $(\xi, id_\mathfrak{g})$. Under this identification, the L_∞-isomorphism given there can be alternatively described as in [7, Corollary A.10] (note that $[\xi \circ c]_\mathfrak{g} = -[\omega_{3p}]_\mathfrak{g}$ as elements of $H^3(\mathfrak{g})$).

The following statement should be compared with Proposition 5.5.3.

Lemma 5.5.10. *Assume $[\omega_{3p}]_\mathfrak{g} \neq 0$. Then $[\omega_{3p}]_{CE(L)} = 0$ iff there exists a surjective L_∞-morphism as in Lemma 5.5.8 a).*

Proof. Let $\eta = \eta|_{s\mathfrak{h}} + \eta|_{S^2 s\mathfrak{g}} \in CE(L)^2$ satisfy $d_{CE(L)}\eta = \omega_{3p}$. The element $\xi := \eta|_{s\mathfrak{h}}$ is non-zero, because $[d_3\xi]_\mathfrak{g} = [\omega_{3p}]_\mathfrak{g} \neq 0$ by Lemma 5.5.2 i). Hence, the L_∞-morphism f from $\mathfrak{h} \oplus \mathfrak{g}$ to $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$ that corresponds to η by Lemma 5.4.8, which satisfies $(f_1)|_\mathfrak{g} = Id_\mathfrak{g}$, has a surjective first component.

For the converse, just apply Lemma 5.4.8. □

The following proposition gives an alternative characterization of the existence of $\mathfrak{h} \oplus \mathfrak{g}$ moment maps.

[4] I.e. the first component is surjective.

[5] An L_∞-isomorphism is an L_∞-morphism whose first component is an isomorphism.

Proposition 5.5.11. *Assume $[\omega_{3p}]_{\mathfrak{g}} \neq 0$. If a $\mathfrak{h} \oplus \mathfrak{g}$ moment map exists, then $\mathfrak{h} \oplus \mathfrak{g}$ has a quotient which is L_∞-isomorphic to $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$ by a morphism that is $Id_{\mathfrak{g}}$ on $\mathfrak{g}$. The converse holds if $H^1(M) = 0$.*

Proof. Combine Corollary 5.4.11, Lemma 5.5.10 and Lemma 5.5.8. □

Example 5.5.12. Suppose that $\mathfrak{g}$ is the Lie algebra of a simple compact Lie group. Let $\mathbb{R} \oplus_{c_e^{\mathfrak{g}}} \mathfrak{g}$ be the string Lie 2-algebra defined in Example 3.4.16, with $c_e^{\mathfrak{g}}$ being the Lie algebra cocycle corresponding to the 3-form $\langle \theta^L, [\theta^L, \theta^L] \rangle$. Then $c_e^{\mathfrak{g}}(x, y, z)$ is given by $\langle x, [y, z] \rangle$, and it generates the 1-dimensional vector space $H^3(\mathfrak{g})$. Assume that $[\omega_{3p}]_{\mathfrak{g}} \neq 0$. Then $[\omega_{3p}]_{\mathfrak{g}}$ and c are multiples of each other, so $\mathbb{R} \oplus_{-\omega_{3p}} \mathfrak{g}$ is L_∞-isomorphic to the string Lie 2-algebra, as one can see, using [7, Cor. A.10]. By Prop. 5.5.11, if a $\mathfrak{h} \oplus \mathfrak{g}$ moment map exists, then necessarily $\mathfrak{h} \oplus \mathfrak{g}$ has the string Lie 2-algebra as a quotient. (The converse holds if $H^1(M) = 0$.)

5.6 Examples

We now present instances in which moment maps for Lie 2-algebras exist, using the explicit criteria developed in §5.5.1 .

In this section $\mathfrak{g}$ is always a Lie algebra, $\mathfrak{h}$ a $\mathfrak{g}$-representation, and c a 3-cocycle for this representation. (We remind that this triple of data is equivalent to a minimal Lie 2-algebra structure on $\mathfrak{h} \oplus \mathfrak{g}$, see Lemma 5.2.4.) Furthermore, (M, ω) is a connected 2-plectic manifold, which we assume to satisfy

$$H^1(M) = 0,$$

and $\mathfrak{g} \to \mathfrak{X}(M), x \mapsto v_x$ is a Lie algebra morphism taking values in Hamiltonian vector fields.

Recall that these data give rise to a reduced Lie 2-algebra as in §5.5.1, corresponding to a triple $(\mathfrak{g}, \mathfrak{h}/[\mathfrak{g}, \mathfrak{h}], c_{red})$. Furthermore, it gives rise to a 3-cocycle ω_{3p} for $\mathfrak{g}$ as in (5.15) (upon an immaterial choice of a point $p \in M$).

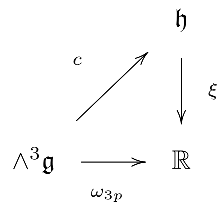
Remark 5.6.1. If $dim(\mathfrak{g}) \leq 2$, then there exists a $\mathfrak{g}$ moment map by Theorem 3.4.13 (since $\wedge^3 \mathfrak{g} = 0$), and thus also a $\mathfrak{h} \oplus \mathfrak{g}$ moment map by Corollary 5.4.1. Therefore, when looking for Lie 2-algebra moment maps that do not arise from Lie algebra moment maps, we should look at Lie algebras with $dim(\mathfrak{g}) \geq 3$.

5.6.1 Abelian Lie algebras

Lemma 5.6.2. *Suppose the Lie algebra* $\mathfrak{g}$ *is abelian. Then* $[\omega_{3p}]_{CE(L)} = 0$ *if and only if* $\omega_{3p}|_U = 0$*, where* $U = \{u \in \wedge^3 \mathfrak{g} : \ c(u) \in [\mathfrak{g}, \mathfrak{h}]\}$.

Remark 5.6.3. In terms of the reduced Lie 2-algebra, the condition $\omega_{3p}|_U = 0$ becomes the following inclusion of kernels of linear maps: $\ker(c_{red} \colon \wedge^3 \mathfrak{g} \to \mathfrak{h}_{red}) \subset \ker(\omega_{3p} \colon \wedge^3 \mathfrak{g} \to \mathbb{R})$.

Proof. Using Proposition 5.5.3 and the fact that $\mathfrak{g}$ is abelian, we deduce that $[\omega_{3p}]_{CE(L)} = 0$ iff there exists $\xi \in [\mathfrak{g}, \mathfrak{h}]^0$ making this diagram commute:



Assume there is such a ξ. Then for $u \in \wedge^3 \mathfrak{g}$ such that $c(u) \in [\mathfrak{g}, \mathfrak{h}]$ we must have $\omega_{3p}(u) = \xi(c(u)) = 0$.

Conversely, if $\omega_{3p}|_U = 0$, then we can define $\xi|_{im(c)}$ by $\xi|_{im(c)}(a) := \omega_{3p}(c^{-1}(a))$ for all $a \in im(c) \subset \mathfrak{h}$, where $c^{-1}(a)$ is any element in the preimage of a under c. Such $\xi|_{im(c)}$ is well-defined, because $\omega_{3p}|_U = 0$ implies that $ker(c) \subset ker(\omega_{3p})$. By defining $\xi|_V = 0$ on any V such that $im(c) \oplus V = \mathfrak{h}$ and extending linearly to the rest of $\mathfrak{h}$, we obtain the desired ξ. □

Using Corollary 5.4.11 we obtain:

Corollary 5.6.4. *Suppose* $\mathfrak{g}$ *is an abelian Lie algebra. Then there is a* $\mathfrak{h} \oplus \mathfrak{g}$ *moment map if and only if* $\omega_{3p}|_U = 0$*, where* $U = \{u \in \wedge^3 \mathfrak{g} : \ c(u) \in [\mathfrak{g}, \mathfrak{h}]\}$.

Example 5.6.5. Consider $(M, \omega) = (\mathbb{R}^3, dx \wedge dy \wedge dz)$. Let the abelian Lie algebra $\mathfrak{g} = \mathbb{R}^3$ act on $M = \mathbb{R}^3$ by translations. This action preserves ω, therefore, the action is generated by multisymplectic, hence Hamiltonian, vector fields. Using Remark 5.6.3 and noticing that $dim(\wedge^3 \mathfrak{g}) = 1$, we can see: *a moment map for* $\mathfrak{h} \oplus \mathfrak{g}$ *exists if and only if* $c_{red} \neq 0$. Here are some concrete simple cases illustrating this result:

- Let $\mathfrak{h}$ be any representation of $\mathfrak{g} = \mathbb{R}^3$, and $c = 0$ the zero cocycle. Since $c_{red} = 0$ in this case, we have no moment map for $(\mathfrak{h} \oplus \mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, c \equiv 0)$.

On the other hand, if $\mathfrak{h}$ is a trivial representation and c is any non-zero cocycle, then $c_{red} \neq 0$, and we have a moment map for $\mathfrak{h} \oplus \mathfrak{g}$.

- For representations $\mathfrak{h}$ of $\mathfrak{g} = \mathbb{R}^3$ such that $[\mathfrak{g}, \mathfrak{h}] = \mathfrak{h}$, there is no moment map for $\mathfrak{h} \oplus \mathfrak{g}$, because $c_{red} = 0$. This is the case, for example, for the representation
$$\mathfrak{g} \to End(\mathfrak{h}), e_i \mapsto \lambda_i Id$$
where the e_i are basis elements of $\mathfrak{g}$, and at least one of the $\lambda_i \in \mathbb{R}$ is non-zero.

5.6.2 Other examples

We present two examples in which the Lie algebra $\mathfrak{g}$ is not abelian. In both of them, just as in Example 5.6.5, the action is given by left translation on a Lie group with vanishing first and second cohomology.

Example 5.6.6 (Connected compact simple Lie groups)**.** As in Example 3.3.8, let G be a connected compact simple Lie group acting on itself by left multiplication. Recall that $H^1(G) = H^2(G) = 0$ (see, e.g., [6]). The Lie algebra $\mathfrak{g}$ of such a group is equipped with a skew non-degenerate trilinear form

$$\theta(x, y, z) := \langle x, [y, z] \rangle$$

called *Cartan 3-cocycle*, where $\langle\ ,\ \rangle$ is the Killing form. Let $\omega = \langle \theta^L, [\theta^L, \theta^L] \rangle$ be as in Example 3.3.8; ω is the left-invariant 2-plectic form on G which equals θ at the identity element e. The action is Hamiltonian, and $[\omega_{3e}]_{\mathfrak{g}} = [\theta]_{\mathfrak{g}} \neq 0$ in $H^3(\mathfrak{g}) \cong \mathbb{R}$.

Thus, if $\mathfrak{h}$ is any representation of $\mathfrak{g}$ and c a 3-cocycle for this representation, Proposition 5.5.5 and Proposition 5.5.7 imply:

There exists a moment map for $\mathfrak{h} \oplus \mathfrak{g}$ *iff* $[c_{red}]_{\mathfrak{g}} \neq 0$.

Note that Remark 5.5.4 applies, and that using the notation introduced there, the condition $[c_{red}]_{\mathfrak{g}} \neq 0$ can be expressed as follows: $[c_i] \neq 0 \in H^3(\mathfrak{g})$ for some $i \in \{1, \ldots, k\}$.

Example 5.6.7 (The Heisenberg Lie algebra)**.** Let $\mathfrak{g}$ be the Lie algebra of the Heisenberg group G, i.e.,

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

Below we will need the following claim: *There is a canonical isomorphism of 1-dimensional vector spaces* $\wedge^3\mathfrak{g}^* \cong H^3(\mathfrak{g})$.

As a smooth manifold, $G \cong \mathbb{R}^3$, hence $H^1(G) = 0$. Let G act on itself by left multiplication, and let ω be a left-invariant volume form: thus, the generators of left translations are multisymplectic and, since $H^2(G) = 0$, Hamiltonian vector fields.

Consider the representation of $\mathfrak{g}$ on $\mathfrak{h} := \mathbb{R}^3$ by matrix multiplication, and let c be any 3-cocycle for this representation. Clearly, $[\mathfrak{g}, \mathfrak{h}] \subsetneq \mathfrak{h}$, and the quotient $\mathfrak{h}/[\mathfrak{g}, \mathfrak{h}]$ is isomorphic to $\mathbb{R}$. We have $[\omega_{3p}]_{\mathfrak{g}} \neq 0$ for any point p, since ω is a volume form, and by the above claim. Since $H^3(\mathfrak{g})$ is 1-dimensional and $[\omega_{3p}]_{\mathfrak{g}} \neq 0$, Proposition 5.5.5, Proposition 5.5.7 and the above claim imply:

There exists a moment map for $\mathfrak{h} \oplus \mathfrak{g}$ *iff* $c_{red} \neq 0$.

To conclude, we prove the above claim. Any $c \in \wedge^3\mathfrak{g}^*$ is closed by dimension reasons, yielding the surjective map $\wedge^3\mathfrak{g}^* \to H^3(\mathfrak{g}), c \mapsto [c]_{\mathfrak{g}}$. This map is injective: for all $\xi \in \wedge^2\mathfrak{g}^*$ we have $d_{\mathfrak{g}}\xi = 0$: using a basis X, Y, Z of $\mathfrak{g}$ satisfying the bracket relations $[X, Y] = Z, [X, Z] = [Y, Z] = 0$, we have

$$(d_{\mathfrak{g}}\xi)(X, Y, Z) = -\xi([X, Y], Z) + \xi([X, Z], Y) - \xi([Y, Z], X) = 0.$$

Bibliography

[1] M. Atiyah. The moment map and equivariant cohomology. *Michael Atiyah Collected Works: Volume 5: Gauge Theories*, 5(1):429, 1988.

[2] J. Baez. Classical mechanics. `http://math.ucr.edu/home/baez/classical/`.

[3] J. C. Baez and A. S. Crans. Higher-dimensional algebra VI: Lie 2-algebras. *Theory Appl. Categ*, 12(15):492–528, 2004.

[4] J. C. Baez, A. E. Hoffnung, and C. L. Rogers. Categorified symplectic geometry and the classical string. Commun.Math.Phys.293:701-725,2010.

[5] V. Bargmann. On unitary ray representations of continuous groups. *Annals of Mathematics*, pages 1–46, 1954.

[6] G. E. Bredon. *Topology and geometry*, volume 139. Springer Science & Business Media, 2013.

[7] M. Callies, Y. Fregier, C. L. Rogers, and M. Zambon. Homotopy moment maps. *Advances in Mathematics*, 303:954–1043, 2016.

[8] A. Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[9] F. Cantrijn, A. Ibort, and M. De León. On the geometry of multisymplectic manifolds. *Journal of the Australian Mathematical Society*, 66(3):303–330, 1999.

[10] J. F. Cariñena, M. Crampin, and L. A. Ibort. On the multisymplectic formalism for first order field theories. *Differential geometry and its Applications*, 1(4):345–374, 1991.

[11] C. Chevalley and S. Eilenberg. Cohomology theory of Lie groups and Lie algebras. *Transactions of the American Mathematical Society*, 63(1):85–124, 1948.

[12] J. A. de Azcárraga and J. M. Izquierdo. *Lie groups, Lie algebras, cohomology and some applications in physics*. Cambridge University Press, 1998.

[13] N. L. Delgado. Lagrangian field theories: ind/pro-approach and L_∞ algebra of local observables. *arXiv preprint arXiv:1805.10317*, 2018.

[14] N. L. Delgado. Multisymplectic structures and higher momentum maps. *arXiv preprint arXiv:1811.01415*, 2018.

[15] T. Delzant. Hamiltoniens périodiques et images convexes de l'application moment. *Bulletin de la Société mathématique de France*, 116(3):315–339, 1988.

[16] M. Doubek, M. Markl, and P. Zima. Deformation theory (lecture notes). *Archivum mathematicum*, 43(5):333–371, 2007.

[17] M. Forger, C. Paufler, and H. Römer. The Poisson bracket for Poisson forms in multisymplectic field theory. *Reviews in Mathematical Physics*, 15(07):705–743, 2003.

[18] Y. Frégier, C. Laurent-Gengoux, and M. Zambon. A cohomological framework for homotopy moment maps. *J. Geom. Phys.*, 97:119–132, 2015.

[19] O. Goertsches and L. Zoller. Equivariant de Rham cohomology: theory and applications. *São Paulo Journal of Mathematical Sciences*, 13(2):539–596, 2019.

[20] M. J. Gotay, J. Isenberg, and J. E. Marsden. Momentum maps and classical relativistic fields. Part II: Canonical analysis of field theories. *arXiv preprint math-ph/0411032*, 2004.

[21] M. J. Gotay, J. Isenberg, J. E. Marsden, and R. Montgomery. Momentum maps and classical relativistic fields. Part I: Covariant field theory. *arXiv preprint physics/9801019*, 1998.

[22] M. J. Gotay and J. A. Isenberg. The symplectization of science. *Gazette des Mathématiciens*, 54:59–79, 1992.

[23] V. Guillemin and S. Sternberg. *Symplectic techniques in physics*. Cambridge university press, 1990.

[24] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[25] G. Heckman and P. Hochs. Geometry of the momentum map. *Proc. Utrecht Spring School*, 2004.

[26] F. Hélein. Multisymplectic formalism and the covariant phase. *arXiv preprint arXiv:1106.2086*, 2011.

[27] F. Hélein and J. Kouneiher. Finite dimensional Hamiltonian formalism for gauge and quantum field theories. *Journal of Mathematical Physics*, 43(5):2306–2347, 2002.

[28] Henry Cohn. Why symplectic geometry is the natural setting for classical mechanics. `https://math.mit.edu/~cohn/Thoughts/symplectic.html`.

[29] J. Herman. Existence and uniqueness of weak homotopy moment maps. *Journal of Geometry and Physics*, 131:52–65, 2018.

[30] J. Herman. Noether's theorem in multisymplectic geometry. *Differential Geometry and its Applications*, 56:260–294, 2018.

[31] C. J. Isham. Topological and global aspects of quantum theory. In *Relativity, groups and topology. 2.* 1984.

[32] N. Jacobson. *Lie algebras.* Number 10. Courier Corporation, 1979.

[33] J. Kijowski. A finite-dimensional canonical formalism in the classical field theory. *Communications in Mathematical Physics*, 30(2):99–128, 1973.

[34] J. Kijowski and W. Szczyrba. Multisymplectic manifolds and the geometrical construction of the poisson brackets in the classical field theory. In *Géométrie symplectique et physique mathématique*, volume 237, pages 347–378. 1975.

[35] A. A. Kirillov. *Lectures on the orbit method*, volume 64. American Mathematical Soc., 2004.

[36] A. W. Knapp. Lie Groups Beyond an Introduction. *Boston, Mass*, 1996.

[37] M. Kontsevich. Deformation quantization of Poisson manifolds. *Letters in Mathematical Physics*, 66(3):157–216, 2003.

[38] Y. Kosmann-Schwarzbach. The Noether theorems. In *The Noether Theorems*, pages 55–64. Springer, 2011.

[39] T. Lada and J. Stasheff. Introduction to SH Lie algebras for physicists. *International Journal of Theoretical Physics*, 32(7):1087–1103, 1993.

[40] J. M. Lee. Smooth manifolds. In *Introduction to Smooth Manifolds*, pages 1–31. Springer, 2013.

[41] T. B. Madsen and A. Swann. Multi-moment maps. *Adv. Math.*, 229(4):2287–2309, 2012.

[42] T. B. Madsen and A. Swann. Closed forms and multi-moment maps. *Geom. Dedicata*, 165:25–52, 2013.

[43] L. Mammadova and L. Ryvkin. On the extension problem for weak moment maps. *arXiv preprint arXiv:2001.00264*, 2020.

[44] L. Mammadova and M. Zambon. Lie 2-algebra moment maps in multisymplectic geometry. *Differential Geometry and its Applications*, 70:101631, 2020.

[45] J. Marsden and A. Weinstein. Reduction of symplectic manifolds with symmetry. *Reports on mathematical physics*, 5(1):121–130, 1974.

[46] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems*, volume 17. Springer Science & Business Media, 2013.

[47] K. R. Meyer. Symmetries and integrals in mechanics. In *Dynamical systems*, pages 259–272. Elsevier, 1973.

[48] R. S. Palais. A global formulation of the Lie theory of transformation groups, Mem. AMS, 1957.

[49] S. M. Pelling. *Multisymplectic methods in field theory with symmetries and BRST*. PhD thesis, King's College London, 2016.

[50] C. L. Rogers. L_∞-algebras from multisymplectic geometry. *Lett. Math. Phys.*, 100(1):29–50, 2012.

[51] C. L. Rogers. 2-plectic geometry, Courant algebroids, and categorified prequantization. *J. Symplectic Geom.*, 11(1):53–91, 2013.

[52] L. Ryvkin. *Observables and Symmetries of N-plectic Manifolds*. Springer, 2016.

[53] L. Ryvkin and T. Wurzbacher. Existence and unicity of co-moments in multisymplectic geometry. *Differential Geom. Appl.*, 41:1–11, 2015.

[54] L. Ryvkin and T. Wurzbacher. An invitation to multisymplectic geometry. *Journal of Geometry and Physics*, 142:9–36, 2019.

[55] L. Ryvkin, T. Wurzbacher, and M. Zambon. Conserved quantities on multisymplectic manifolds. *Journal of the Australian Mathematical Society*, 108(1):120–144, 2020.

[56] H. Sati, U. Schreiber, and J. Stasheff. L_∞-algebra connections and applications to String-and Chern-Simons n-transport. In *Quantum field theory*, pages 303–424. Springer, 2009.

[57] J.-M. Souriau. Structure of dynamical systems, volume 149 of Progress in Mathematics. *A Symplectic View of Physics*, 1997.

[58] J. Stasheff. L_∞ and A_∞-structures: then and now. *arXiv preprint arXiv:1809.02526*, 50, 1809.

[59] L. Vitagliano. Strong homotopy structures from foliations. 2012.

[60] L. Vitagliano. On the strong homotopy Lie–Rinehart algebra of a foliation. *Communications in Contemporary Mathematics*, 16(06):1450007, 2014.

[61] D. Vogan. Review of "Lectures on the orbit method" by A. A. Kirillov. *Bulletin of the AMS*, 1997.

[62] Z. Wang. Introduction to Lie groups. `http://staff.ustc.edu.cn/~wangzuoq/Courses/13F-Lie/Lie.html#info`, 2013.

[63] Z. Wang. Symplectic geometry. `http://staff.ustc.edu.cn/~wangzuoq/Courses/15S-Symp/SympGeom.html`, 2015.

[64] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[65] A. Weinstein. Symplectic geometry. *Bulletin of the American mathematical society*, 5(1):1–13, 1981.

[66] N. M. J. Woodhouse. *Geometric quantization*. Oxford University Press, 1997.

[67] M. Zambon. L_∞-algebras and higher analogues of Dirac structures and Courant algebroids. *J. Symplectic Geom.*, 10(4):563–599, 2012.

FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
GEOMETRY SECTION
Celestijnenlaan 200B box 2400
B-3001 Leuven
leyli.mammadova@kuleuven.be
http://wis.kuleuven.be/meetkunde